# The finite gap ansatz for the semiclassical focusing NLS equation with bell-shaped initial data

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# The name of the game...

In this talk we are interested in the finite gap ansatz for the semiclassical focusing $(1+1)$-dimensional nonlinear Schrödinger equation with cubic nonlinearity and bell-shaped initial data of some smoothness that decay rapidly. Our ultimate aim is to arrive at the formula that gives us the semiclassical behavior of the solutions of this initial value problem and comment on that. Particularly, this formula is given in terms of theta functions on some associated Riemann surfaces that have to do with the study of some corresponding equilibrium measure problems.

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# The classical NLS equation

## The IVP for the $(1+1)$ focusing cubic NLS

Fix $\hbar > 0$ and consider for the complex field $u$ the problem

$$\begin{cases} i\hbar\partial_t u + \frac{\hbar^2}{2}\partial_x^2 u + |u|^2 u = 0, & (x,t) \in \mathbb{R} \times [0, +\infty) \\ u(x,0) = A(x), & x \in \mathbb{R}. \end{cases}$$

where $A(x)$ is a function of some smoothness, that tends rapidly to zero for large $|x|$.

# The classical NLS equation

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How does a solution to such an IVP look like?

# Numerics for NLS: depicting a solution

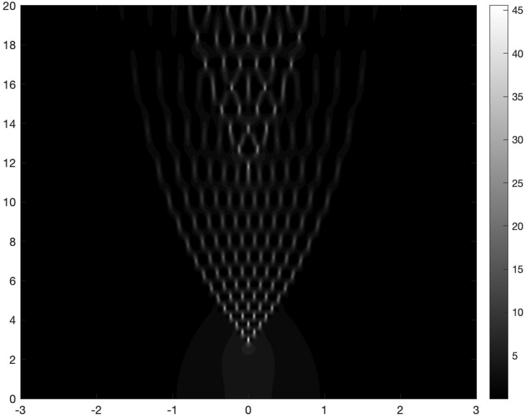


Figure: The numerically calculated solution of the IVP with $\hbar = 0.1$ and $u(x,0) = \text{sech}(2x)$.The squared modulus $|u|^2(x,t)$ is shown in this plot.

The horizontal axis represents $x$ while $t$ runs on the vertical axis as shown on the left side of the figure. The bar on the right of the plot shows a graduated scale for the values of $|u|^2$ ranging from deep black (where $|u|^2$ is small) to bright white (where $|u|^2$ is big).

# Semiclassical f-NLS with cubic nonlinearity & fast decaying initial data

## The semiclassical NLS initial value problem ($\hbar$ no more fixed)

For the complex field $u$, consider the problem

$$\begin{cases} i\hbar\partial_t u + \frac{\hbar^2}{2}\partial_x^2 u + |u|^2 u = 0, \quad (x,t,\hbar) \in \mathbb{R}\times[0,+\infty)\times(0,+\infty) \\ u(x,0,\hbar) = A(x), \quad (x,\hbar)\in\mathbb{R}\times(0,+\infty). \end{cases}$$

where $A(x)$ is a function satisfying

$$(A): \begin{cases} A(x) > 0, \ x\in\mathbb{R} \\ A(-x) = A(x), \ x\in\mathbb{R} \\ \text{has one single local maximum at } x=0 \text{ where } A(0) = A_{\max} \\ A''(0) < 0 \\ A(x) \text{ is a Schwartz function.} \end{cases}$$

# Question & Applications

## A semiclassical question

Suppose that $\hbar$ is small compared to the $x, t$. What is the behavior of solutions of the problem above as $\hbar \downarrow 0$?

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Some applications where one finds problems like the previous:

- Propagation of light in nonlinear optical fibers (Manakov system, spatial/temporal solitons, dark solitons): The equation describes the propagation of the wave through the nonlinear medium.
- Small-amplitude gravity waves on the surface of deep inviscid water (rogue waves, rogue holes, Peregrine soliton/breather): It describes the evolution of the envelope of modulated wave groups; the depth of the water is large compared to the wave length of the water waves.
- Bose-Einstein condensates (e.g.Thomas-Fermi limits)

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# Scattering data

Zakharov and Shabat (1972) proved that the (classical) NLS can be integrated via the so-called Inverse Scattering Transform (IST). The main ingredient of the method is the notion of scattering data $s(t, \hbar)$.

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At each time $t \geq 0$ there is some $N(\hbar) \in \mathbb{N}$ independent of $t$ so that

$$s(t, \hbar) = \left\{ \{\lambda_n(\hbar)\}_{n=1}^{N(\hbar)}, \{\kappa_n(t, \hbar)\}_{n=1}^{N(\hbar)}, R(\cdot, t, \hbar) \right\}$$

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$$s(t, \hbar) = \left\{ \{\lambda_n(\hbar)\}_{n=1}^{N(\hbar)}, \{\kappa_n(t, \hbar)\}_{n=1}^{N(\hbar)}, R(\cdot, t, \hbar) \right\}$$

where

- $\lambda_n(\hbar) \in \mathbb{C}$ represent the (time independent) eigenvalues of the Lax operator $\mathfrak{D}_\hbar$
- $\kappa_n(t, \hbar)$ are their corresponding norming constants (the proportionality constants for the corresponding eigenfunctions) and
- $R(\cdot, t, \hbar) : \sigma_c(\mathfrak{D}_\hbar) \to \mathbb{C}$ is a function called the reflection coefficient (it quantifies the phenomenon of reflection that a wave undergoes in the presence of the potential)

# The IST

We arrive at the desired formula for the semiclassical behavior of the solution, by utilizing the IST for the NLS. The whole procedure can be described by the following scheme:

$$u(\cdot, 0, \hbar) = A(\cdot) \quad \xrightarrow[\text{Lax pair}]{\text{forward scattering problem}} \quad s(0, \hbar)$$

$$\Big\downarrow \ \text{ODE}$$

$$u(\cdot, t, \hbar) \quad \xleftarrow[\text{Riemann-Hilbert problems}]{\text{inverse scattering problem}} \quad s(t, \hbar)$$

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# Notation (I)

Throughout this talk, we use the following notation:

- $\mathbb{H} = \{z \in \mathbb{C} \mid \Im z > 0\}$ and $\overline{\mathbb{H}}$ its closure in $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$
- $\mathbb{K} = \{z \in \mathbb{C} \mid \Im z > 0\} \setminus \{z \in \mathbb{C} \mid \Re z = 0 \text{ and } 0 < \Im z \leq A_{\max}\}$ and $\overline{\mathbb{K}}$ its closure in $\overline{\mathbb{C}}$
- On the segment $[0, iA_{\max}] \subset \overline{\mathbb{K}}$: For $0 \leq x < A_{\max}$, we consider the points $ix_+$ and $ix_-$ as distinct. In other words, we cut a slit in the upper half-plane along the segment $[0, iA_{\max}]$ and distinguish between the two sides of the slit
- $\mathbb{F} = \{F \subset \overline{\mathbb{K}} \mid F \text{ is connected \& compact containing } 0_+ \text{ and } 0_-\}$ Such an $F \in \mathbb{F}$ is called a **continuum**.

# Notation (II)

Let $\rho^0(z)$ be a given complex-valued function on $\overline{\mathbb{H}}$ satisfying the following condition

$$(B): \begin{cases} \rho^0(z) \text{ is holomorphic in } \mathbb{H} \\ \rho^0(z) \text{ is continuous in } \overline{\mathbb{H}} \\ \Re[\rho^0(z)] = 0, \quad z \in [0, iA_{\max}] \\ \Im[\rho^0(z)] > 0, \quad z \in (0, iA_{\max}] \cup \mathbb{R}. \end{cases}$$

Consider **Green's function for the upper half-plane**

$$G(z;\eta) = log \frac{|z-\eta^*|}{|z-\eta|}$$

where the star superscript denotes complex conjugation and let $\mu^0$ be the non-negative measure on the segment $[0, iA_{\max}]$ (oriented from 0 to $iA_{\max}$) defined by

$$d\mu^0(\eta) = -\rho^0(\eta)d\eta.$$

# Notation (III)

Now, let $\phi$ be defined for $z \in \overline{\mathbb{K}}$ by

## Our external field

$$\phi(z;x,t) = -\int_{[0,iA_{\max}]} G(z;\eta)d\mu^0(\eta) + \Re\left[iJ(x)\left(\pi\int_{[0,iA_{\max}]} d\mu^0(\eta) - 2zx + 2z^2t\right)\right]$$

where $(x,t) \in \mathbb{R} \times [0,+\infty)$. Here $x,t$ are viewed as parameters and in fact represent the space and time variables of our NLS IVP. Also

$$J(x) = \begin{cases} 1 & \text{for } x \geq 0 \\ -1 & \text{for } x < 0. \end{cases}$$

# Notation (IV)

Now, let $\mathcal{B}_+(\overline{\mathbb{K}})$ be the set of positive Borel measures on $\overline{\mathbb{K}}$.

## The free energy of a measure

For $\mu \in \mathcal{B}_+(\overline{\mathbb{K}})$ we define

$$E[\mu] = \int\int_{\overline{\mathbb{K}}\times\overline{\mathbb{K}}} G(z;\eta)d\mu(z)d\mu(\eta).$$

Next, define the set of measures

$$\mathcal{M}(\overline{\mathbb{K}}) = \left\{ \mu \in \mathcal{B}_+(\overline{\mathbb{K}}) \,\middle|\, E[\mu] \,\&\, \int_{\overline{\mathbb{K}}} \phi(z;x,t)d\mu(z) \text{ are finite} \right\}.$$

# Notation (V)

## Green's potential of a measure

For $\mu \in \mathcal{M}(\overline{\mathbb{K}})$ let

$$V^{\mu}(z) = \int_{\overline{\mathbb{K}}} G(z;\eta)d\mu(\eta), \ \ z \in \overline{\mathbb{K}}$$

## Weighted energy of a measure

For $\mu \in \mathcal{M}(\overline{\mathbb{K}})$ let

$$E_{\phi}[\mu](x,t) = E[\mu] + 2\int_{\overline{\mathbb{K}}} \phi(z;x,t)d\mu(z)$$

# Notation (VI)

Now, for any continuum $F \in \mathbb{F}$, define the set of measures

$$\mathcal{M}_F(\overline{\mathbb{K}}) = \{\mu \in \mathcal{M}(\overline{\mathbb{K}}) \,|\, \mathrm{supp}(\mu) \subseteq F\}$$

## Equilibrium measure supported in $F$

If there exists a measure in $\mathcal{M}_F(\overline{\mathbb{K}})$ that minimizes the weighted energy with respect to the field $\phi$, then we shall call it the **equilibrium measure with respect to the field $\phi$, supported in $F$**. We will denote it by $\lambda^F$. So that

$$E_\phi[\lambda^F] = \min_{\mu \in \mathcal{M}_F(\overline{\mathbb{K}})} E_\phi[\mu]$$

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# An important theorem

## Theorem [existence of an S-curve for every point (x,t)]

For the above external field $\phi$ with $(x,t) \in \mathbb{R} \times [0, +\infty)$, there exists a continuum $C_{x,t} \in \mathbb{F}$ such that the equilibrium measure $\lambda^{C_{x,t}}$ exists and

$$E_\phi[\lambda^{C_{x,t}}] = \max_{F \in \mathbb{F}} E_\phi[\lambda^F] = \max_{F \in \mathbb{F}} \min_{\mu \in \mathcal{M}_F(\overline{\mathbb{K}})} E_\phi[\mu].$$

Furthermore, $C_{x,t}$ is a piecewise smooth contour, such that $\mathrm{supp}(\lambda^{C_{x,t}})$ consists of a union of finitely many analytic arcs and on each interior point of $\mathrm{supp}(\lambda^{C_{x,t}})$ we have

$$\frac{d}{dn_+}(\phi + V^{\lambda^{C_{x,t}}}) = \frac{d}{dn_-}(\phi + V^{\lambda^{C_{x,t}}})$$

where the two derivatives in this formula denote the normal derivatives [to $\mathrm{supp}(\lambda^{C_{x,t}})$]. Such a curve $C_{x,t}$ shall be called **S-curve**.

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# G-function

Fix a point $(x, t) \in \mathbb{R} \times [0, +\infty)$ and let $C_{x,t}$ be a maximizing contour guaranteed by the previous theorem. Denote by $C^*_{x,t}$ its complex-conjugate symmetric. A priori, we seek a function $g(\lambda; x, t)$ satisfying the following

$$(C): \begin{cases} g(\lambda; x, t) \text{ is independent of } \hbar \\ g(\lambda; x, t) \text{ is analytic for } \lambda \in \mathbb{C} \setminus (C_{x,t} \cup C^*_{x,t}) \\ g(\lambda; x, t) \to 0 \text{ as } \lambda \to \infty \\ g(\lambda; x, t) \text{ assumes cont. bd values from both sides of } C_{x,t} \cup C^*_{x,t} \\ g(\lambda^*; x, t) + g(\lambda; x, t)^* = 0 \text{ for all } \lambda \in \mathbb{C} \setminus (C_{x,t} \cup C^*_{x,t}). \end{cases}$$

The assumptions above are satisfied if we write $g(\lambda; x, t)$ in terms of the maximizing equilibrium measure of the previous theorem, $d\lambda^{C_{x,t}}(\eta) = \rho(\eta; x, t) d\eta$ Indeed,

$$g(\lambda; x, t) = \int_{C_{x,t} \cup C^*_{x,t}} log(\lambda - \eta) \rho(\eta; x, t) d\eta$$

for an appropriate definition of the logarithmic branch.

# G-function (II)

## The functions $\theta$ and $\Phi$

For $\lambda \in C_{x,t} \cup C^*_{x,t}$ define the functions

$$\theta(\lambda; x,t) = iJ(x)(g_+(\lambda; x,t) - g_-(\lambda; x,t))$$

$$\begin{aligned}\Phi(\lambda; x,t) = &\int_{[0, iA_{\max}]} log(\lambda - \eta)\rho^0(\eta)\, d\eta \\ &+ \int_{[-iA_{\max}, 0]} log(\lambda - \eta)\rho^0(\eta^*)^*\, d\eta \\ &+ J(x)(2i\lambda x + 2i\lambda^2 t + i\pi \int_{[\lambda, iA_{\max}]} \rho^0(\eta)\, d\eta \\ &- g_+(\lambda; x,t) - g_-(\lambda; x,t))\end{aligned}$$

where $g_+, g_-$ denote the boundary values of $g(\cdot; x,t)$ from the left and right sides of $C_{x,t} \cup C^*_{x,t}$ respectively.

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# Finite Genus Ansatz (I)

The finite genus ansatz implies that for each $(x, t) \in \mathbb{R} \times [0, +\infty)$ there is a finite positive even integer $G \equiv G_{x,t}$ such that the contour $C \equiv C_{x,t}$ can be divided into **bands** (the support of the equilibrium measure) and **gaps**

- The bands: We denote the bands by $I_j$. More precisely, we define the analytic arcs $I_j, I_j^*$, $j = 0, ..., G/2$ as follows (they come in conjugate pairs). Let the points $\lambda_j$, $j = 0, ..., G$, in the open upper half-plane be the branch points of the function $g(\cdot; x, t)$. All such points lie on the contour $C$ and we order them as $\lambda_0, \lambda_1, ..., \lambda_G$, according to the direction given to $C$. The points $\lambda_0^*, \lambda_1^*, ..., \lambda_G^*$ are their complex conjugates. Then let $I_0 = C[0 \rightsquigarrow \lambda_0]$ be the subarc of $C$ joining points 0 and $\lambda_0$ (in the direction befined by $C$). Similarly, $I_j = C[\lambda_{2j-1} \rightsquigarrow \lambda_{2j}]$, $j = 1, ..., G/2$.
- The gaps: The connected components of the set $\mathbb{C} \setminus \cup_{j=0}^{G/2}(I_j \cup I_j^*)$ are the so-called gaps. For example the gap $\Gamma_1$ is $C(\lambda_0 \rightsquigarrow \lambda_1)$, meaning the subarc of $C$ joining the points $\lambda_0$ and $\lambda_1$ excluding these very endpoints. The rest of the gaps are defined similarly.

It actually follows from the properties of $g, \rho$ that the function $\theta(\lambda; x, t)$ defined for $\lambda \in C_{x,t}$ is constant (in $\lambda$ but not in $x, t$) on each of the gaps $\Gamma_j$, taking a value which we will denote by $\theta_j$, while the function $\Phi(\lambda; x, t)$ defined for $\lambda \in C_{x,t}$ is constant (in $\lambda$ but not in $x, t$) on each of the bands, taking the value denoted by $\Phi_j$ on the band $I_j$. We write this explicitly

## Behavior of $\theta$ on gaps and $\Phi$ on bands

We have

$$\theta(\lambda; x, t) = \theta_j(x, t), \quad \lambda \in \Gamma_j, \quad j = 1, ..., \tfrac{G}{2}, \tfrac{G}{2} + 1$$
$$\Phi(\lambda; x, t) = \Phi_j(x, t), \quad \lambda \in I_j, \quad j = 0, 1, ..., \tfrac{G}{2}.$$

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# Construction

For our purpose, fix a point $(x,t) \in \mathbb{R} \times [0,+\infty)$ and construct a Riemann surface $\mathcal{S} \equiv \mathcal{S}_{x,t}$ by cutting two copies of the complex sphere, namely $\mathcal{S}^+$ and $\mathcal{S}^-$, along the slits $I_0 \cup I_0^*,\ I_j,\ I_j^*,\ j = 1....,G$, and pasting the "top" copy $\mathcal{S}^+$ to the "bottom" copy $\mathcal{S}^-$ along these very slits. The *moduli* of $\mathcal{S}$ are given by $\lambda_j,\ j = 0,....,G$ and their complex conjugates $\lambda_j^*,\ j = 0,....,G$. The *genus* of $\mathcal{S}$ is $G$.

# Holmology cycles

We define the homology cycles $a_j, b_j,\ j = 1, ..., G$. For the $a$-cycles we have

- cycle $a_1$ goes around the slit $l_0 \cup l_0^*$ (that joins $\lambda_0$ to $\lambda_0^*$), remaining on the top sheet and oriented counterclockwise
- cycle $a_2$ goes through the slits $l_1^*$ and $l_1$ starting from the top sheet, also oriented counterclockwise
- cycle $a_3$ goes around the slits $l_1^*$, $l_0 \cup l_0^*$, $l_1$ remaining on the top sheet, oriented counterclockwise
- the rest of the $a$-cycles are defined accordingly

For the $b$-cycles we have

- cycle $b_1$ goes through $l_0 \cup l_0^*$ and $l_1$, oriented counterclockwise
- cycle $b_2$ goes around $l_0 \cup l_0^*$ and $l_1$, remaining on the top sheet, and oriented counterclockwise
- cycle $b_3$ goes through $l_1^*$ and $l_2$, and oriented counterclockwise
- the rest of the $b$-cycles are defined accordingly.

# Holomorphic differentials (I)

First, let us introduce the function $R(\lambda)$ defined by

$$R(\lambda)^2 = \prod_{k=0}^{G} (\lambda - \lambda_k)(\lambda - \lambda_k^*),$$

choosing the particular branch that is cut along the bands $I_k^+$ and $I_k^-$ and satisfies

$$\lim_{\lambda \to \infty} \frac{R(\lambda)}{\lambda^{G+1}} = -1.$$

This defines a real function, i.e. one that satisfies $R(\lambda^*) = R(\lambda)^*$. At the bands, we have $R_+(\lambda) = -R_-(\lambda)$, while $R(\lambda)$ is analytic in the gaps.

# Holomorphic differentials (II)

On $\mathcal{S}$ there is a complex $G$-dimensional linear space of holomorphic differentials, with basis elements $\nu_k(P)$ for $k = 1, \ldots, G$ that can be written in the form

$$\nu_k(P) = \frac{\sum_{j=0}^{G-1} c_{kj} \lambda(P)^j}{R_{\mathcal{S}}(P)} \, d\lambda(P)$$

where $R_{\mathcal{S}}(P)$ is a "lifting" of the function $R(\lambda)$ from the cut plane to $\mathcal{S}$, i.e.

$$R_{\mathcal{S}}(P) = \begin{cases} R(\lambda(P)), & \text{if } P \in \mathcal{S}^+ \\ -R(\lambda(P)), & \text{if } P \in \mathcal{S}^-. \end{cases}$$

# Holomorphic differentials (III)

The coefficients $c_{kj}$ are uniquely determined by the constraint that the differentials satisfy the normalization conditions

$$\oint_{a_j} \nu_k(P) = 2\pi i \delta_{jk}.$$

## The period matrix

The matrix $H$ with elements

$$H_{jk} = \oint_{b_j} \nu_k(P).$$

is called period matrix (it is symmetric with negative definite real part).

# Theta functions

We can define the theta function

$$\Theta(\mathbf{w}) = \sum_{\mathbf{n}\in\mathbb{Z}^G} \exp\left(\frac{1}{2}\mathbf{n}^T H\mathbf{n} + \mathbf{n}^T\mathbf{w}\right), \quad \mathbf{w} \in \mathbb{C}^G$$

where $H$ is the period matrix associated to $\mathcal{S}$ (since the real part of $H$ is negative definite, the series converges).

# Abel map

We define the *lattice* of $\mathcal{S}$, to be the following subset of $\mathbb{C}^G$

$$\Lambda(\mathcal{S}) = \{\mathbf{m} + H\mathbf{n} \,|\, \mathbf{m}, \mathbf{n} \in \mathbb{Z}^G\}.$$

With this in hand, we can define the *Jacobian variety* of $\mathcal{S}$ as

$$\mathcal{J}(\mathcal{S}) = \mathbb{C}^G / \Lambda(\mathcal{S})$$

which by standard theory we know that it is a compact, commutative, $G$-dimensional, complex Lie group.

Now, we arbitrarily fix a base point $P_0 \in \mathcal{S}$. The Abel map with base point $P_0$, i.e. $A(\cdot; P_0) : \mathcal{S} \to \mathcal{J}(\mathcal{S})$, is then defined as follows

$$A(P; P_0) = \left\langle \left[ \int_{P_0}^{P} \nu_1(P'), \ldots, \int_{P_0}^{P} \nu_G(P') \right]^T \right\rangle$$

where $P'$ is an integration variable and $< \mathbf{z} > \in \mathcal{J}(\mathcal{S})$ represents the equivalence class of the element $\mathbf{z} \in \mathbb{C}^G$ under the equivalence relation in $\mathcal{J}(\mathcal{S})$. We also define $A_k(P; P_0)$, $k = 1, \ldots, G$ to be the $k^{\text{th}}$ component of $A(P; P_0)$.

# Meromorphic differentials (I)

Let us first introduce the following Riemann-Hilbert problem. Find a function $\kappa(\lambda) \equiv \kappa(\lambda; x, t)$ such that it satisfies the jump relations

$$\kappa_+(\lambda) - \kappa_-(\lambda) = -\frac{\theta_n}{R(\lambda)}, \quad \lambda \in {\Gamma_n}^+ \cup {\Gamma_n}^-, \quad n = 1, \ldots, \tfrac{G}{2}, \tfrac{G}{2}+1$$

$$\kappa_+(\lambda) - \kappa_-(\lambda) = -\frac{\Phi_n}{R_+(\lambda)}, \quad \lambda \in {I_n}^+ \cup {I_n}^-, \quad n = 0, 1, \ldots, \tfrac{G}{2}$$

and is otherwise analytic (here, ${\Gamma_n}^\pm, {I_n}^\pm \in \mathcal{S}^\pm$). It blows up like $(\lambda - \lambda_n)^{-1/2}$ near each endpoint, has continuous boundary values in between the endpoints, and vanishes like $1/\lambda$ for large $\lambda$.

# Meromorphic differentials (II)

It can be shown that this problem has a unique solution that satisfies

$$\kappa(\lambda) = \frac{1}{2\pi i} \sum_{n=1}^{\frac{G}{2}+1} \theta_n \int_{\Gamma_n{}^+ \cup \Gamma_n{}^-} \frac{d\eta}{(\lambda-\eta)R(\eta)} + \frac{1}{2\pi i} \sum_{n=0}^{\frac{G}{2}} \Phi_n \int_{I_n{}^+ \cup I_n{}^-} \frac{d\eta}{(\lambda-\eta)R_+(\eta)} .$$

# Meromorphic differentials (III)

Next let

$$H(\lambda) = \kappa(\lambda) R(\lambda).$$

The factor $R(\lambda)$ on the formula above, renormalizes the singularities of $\kappa$ at the endpoints, so that the boundary values of $H(\lambda)$ are bounded continuous functions. Near infinity, we have the asymptotic expansion

$$\begin{aligned} H(\lambda) &= H_G \lambda^G + H_{G-1} \lambda^{G-1} + \cdots + H_1 \lambda + H_0 + \mathcal{O}(\lambda^{-1}) \\ &\equiv p(\lambda) + \mathcal{O}(\lambda^{-1}) \end{aligned}$$

where all coefficients $H_j$, $j = 0, 1, \ldots G$ of the polynomial $p(\lambda)$ can be found explicitly by expanding $R(\lambda)$ and the Cauchy integral $\kappa(\lambda)$, for large $\lambda$. It is easy to see from the reality of $\theta_j$ and $\Phi_j$ that $p(\lambda)$ is a polynomial with real coefficients.

# Meromorphic differentials (IV)

Next, we define a certain meromorphic differential on $\mathcal{S}$. Let $\Omega(P)$ be holomorphic away from the infinity points $\infty^{\pm} \in \mathcal{S}$ (where $\infty^{\pm}$ represents the infinity point on the sheet $\mathcal{S}^{\pm}$), where it has the behavior

$$\Omega(P) = \pm dp(\lambda(P)) + \mathcal{O}\left(\frac{d\lambda(P)}{\lambda(P)^2}\right), \quad P \to \infty^{\pm}$$

and made unique by the normalization conditions

$$\oint_{a_j} \Omega(P) = 0, \quad j = 1, \ldots, G.$$

Note that $\Omega(P)$ has no residues.

# Aux parameters

Let the vector $\mathbf{U} \in \mathbb{C}^G$ be defined by

$$\mathbf{U} = \left[ \oint_{b_1} \Omega(P), \ldots, \oint_{b_G} \Omega(P) \right]^T$$

and consider the following two vectors in $\mathbb{C}^G$

$$\mathbf{k} = \partial_x \mathbf{U}$$
$$\mathbf{w} = -\partial_t \mathbf{U}.$$

# Aux parameters (II)

Also, let the vectors $\mathbf{V}, \mathbf{W} \in \mathbb{C}^G$ be defined componentwise by

$$V_k = A_k(\lambda_1^{*+}; \lambda_0{}^+) + A_k(\lambda_2{}^+; \lambda_0{}^+) + A_k(\lambda_3^{*+}; \lambda_0{}^+) + \cdots + A_k(\lambda_G{}^+; \lambda_0{}^+) + A_k(\infty^+; \lambda_0{}^+) + \pi i + \frac{H_{kk}}{2}$$

$$W_k = A_k(\lambda_1^{*+}; \lambda_0{}^+) + A_k(\lambda_2{}^+; \lambda_0{}^+) + A_k(\lambda_3^{*+}; \lambda_0{}^+) + \cdots + A_k(\lambda_G{}^+; \lambda_0{}^+) - A_k(\infty^+; \lambda_0{}^+) + \pi i + \frac{H_{kk}}{2}$$

for $k = 1, ..., G$. It should be emphasized that the $+$ superscripts for the arguments in the $k^{\mathrm{th}}$ component $A_k$ of the Abel map, stand for the corresponding points on the top sheet $\mathcal{S}^+$ of our Riemann surface $\mathcal{S}$. And so, integration occurs on the top sheet as well.

# Aux parameters (III)

After setting

$$\mathbf{Y} = -A(\infty^+; \lambda_0{}^+) - \mathbf{V} \in \mathbb{C}^G$$
$$\mathbf{Z} = \ \ A(\infty^+; \lambda_0{}^+) - \mathbf{W} \in \mathbb{C}^G$$

we finally define the scalars

$$a = \frac{\Theta(\mathbf{Z})}{\Theta(\mathbf{Y})} \sum_{k=0}^{G} (-1)^k \Im(\lambda_k)$$
$$U_0 = -\theta_1 - \Phi_0$$
$$k_0 = \ \partial_x U_0$$
$$w_0 = -\partial_t U_0$$

We simply note here that the $U_i$, $i = 0, \ldots, G$ are real (the same stands for the $k_i$ and $w_i$ $i = 0, \ldots, G$).

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# At last we arrrived at the end...

## Semiclassical asymptotics

Let $(x_0, t_0)$ be any given point so that $x_0 \in \mathbb{R}$ and $t_0 > 0$. The solution $u$ of our IVP is asymptotically ($\hbar \downarrow 0$) described (locally) as a slowly modulated $G$ phase wavetrain. More precisely, setting $x = x_0 + \hbar\hat{x}$ and $t = t_0 + \hbar\hat{t}$, the solution

$$u(x,t) = u(x_0 + \hbar\hat{x}, t_0 + \hbar\hat{t}\,)$$

has the following leading order asymptotics as $\hbar \downarrow 0$

$$u(x,t) \sim a(x_0,t_0)\exp\left\{ i\left( \frac{U_0(x_0,t_0)}{\hbar} + k_0(x_0,t_0)\hat{x} - w_0(x_0,t_0)\hat{t}\right)\right\} \cdot \frac{\Theta\left(\mathbf{Y}(x_0,t_0) + i\left(\frac{\mathbf{U}(x_0,t_0)}{\hbar} + \mathbf{k}(x_0,t_0)\hat{x} - \mathbf{w}(x_0,t_0)\hat{t}\right)\right)}{\Theta\left(\mathbf{Z}(x_0,t_0) + i\left(\frac{\mathbf{U}(x_0,t_0)}{\hbar} + \mathbf{k}(x_0,t_0)\hat{x} - \mathbf{w}(x_0,t_0)\hat{t}\right)\right)}.$$

# Comments on the previous asymptotics

## Some remarks

- We note that the denominator of the approximant in the formula above, never vanishes (for any $x_0, t_0, \hat{x}, \hat{t}$).
- We do not know if the support of the equilibrium measure of the maximizing continuum is unique. But the asymptotic formula depends only on the endpoints $\lambda_j$ of the analytic subarcs of the support. Since this asymptotic expression must be unique, it is easy to see that the endpoints also must be unique.
- Different Riemann surfaces give different formulae (except of course in degenerate cases: a degenerate genus 2 surface can be a pinched genus 0 surface and so on).

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# Relevant works

For all the details, take a look at:

- S. Kamvissis, K. D. T. R. McLaughlin and P. D. Miller, *Semiclassical Soliton Ensembles for the Focusing Nonlinear Schrödinger Equation* , Annals of Mathematics 154 (2003), Princeton University Press, Princeton, NJ.
- S. Kamvissis, E. A. Rakhmanov, *Existence and Regularity for an Energy Maximization Problem in Two Dimensions* , Journal of Mathematical Physics 46, no. 8 (2005)

THANK YOU!