Fluid-Structure Interaction Algorithms

Modeling, Mathematical Setting and Algorithm

Olivier Pironneau[1]

[1] with **Tomas Chacon, Vivette Girault, Frédéric Hecht & Franņois Murat**
University of Paris VI (UPMC), Laboratoire J.-L. Lions (LJLL), **Olivier.Pironneau@upmc.fr**

Conference dedicate to Academician G.I. Marchuk

Plan

- Multi-fluids method (fictitious domain)
- Immersed Boundary method
- Fluid-structure interaction with ALE for the fluid
- Generalities about decomposition algorithms
- Fluid-structure interaction on a fixed domain my own contribution
- An inverse problem

Disclaimer

- Very large field, difficult to read everything
- Some in the audience are more qualified than me
- I'll be happy to improve my knowledge especially on Russian work on FSI

Currently Useful Applications

Goals: fast and meaningful computations

- Aerospace: wing deformation, rocket reservoir ...
- Hemodynamics: blood flows in heart, vessels...
- Rubber and Fluids: tires, shock absorbers
- Swimming: fish, micro-organism

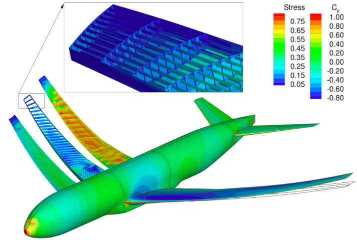
Modeling

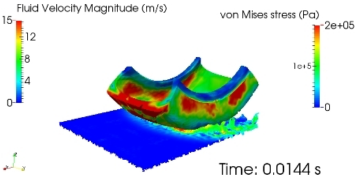
- Everything is one deformable solid (Gonzalez, Simo, LeTallec...)
- Everything is one viscous compressible fluid (Peskin, Gastaldi, Coupez ...)
- Fluid-Structure Interaction with large displacement
- Fluid-Structure (Shell) Interaction with small displacement

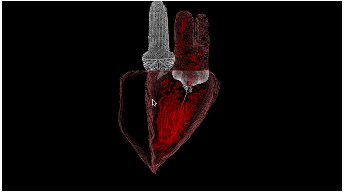
Results

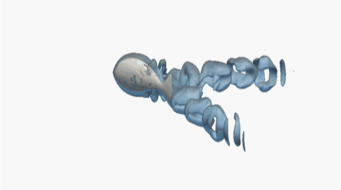
Johachim Martin et al (U of Michigan)

INSA - Université de Lyon









B. Griffith, C. Peskin et al.

M. Bergman and A. Iollo

All Fluids Approach by penalisation

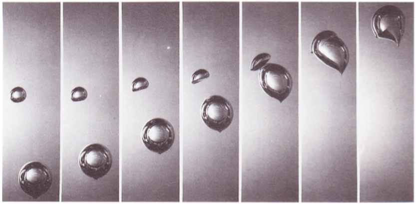
In the solid $(\rho^s, \mu^s) >> (\rho^f, \mu^f)$

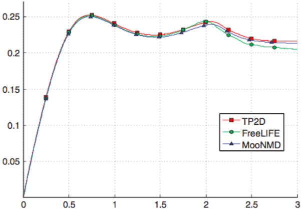
$$\rho(\partial_t u + u \cdot \nabla u) - \nabla \cdot (2\mu D(u) - pI) = \rho \vec{g} + f, \quad \nabla \cdot u = 0$$

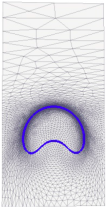
- The interface can be tracted by a level set $\partial_t \phi + u \nabla \phi = 0$ and

$$\rho = \rho^f \mathbf{1}_{x:\phi(x)<-\epsilon} + \rho^s \mathbf{1}_{x:\phi(x)>\epsilon} + (\rho^s - \rho^f)(1 + \frac{\phi}{\epsilon} + \frac{1}{\pi} \sin \frac{\phi}{\epsilon}) \mathbf{1}_{x:-\epsilon \le \phi(x) \le \epsilon}.$$

- At any interface there is continuity of velocity and normal stress built in.
- Surface tension is $f = \sigma \kappa \vec{n} \delta_{x \in S}$ with $\vec{n} = \dfrac{\nabla \phi}{|\nabla \phi|}$ and $\kappa = -\nabla \cdot \vec{n}$, $\epsilon = O(\sqrt{h})$

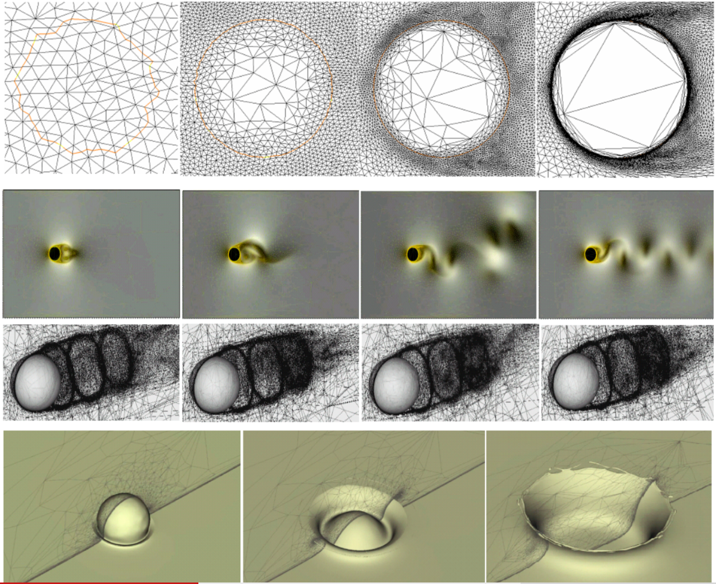






A gallery of fluid motions (2003) Velocity vs time of a rising bubble with $(\rho^s, \mu^s) = 1000(\rho^f, \mu^f)$ (Hysing-Yamaguchi-Otsuka-Marrouf-Th. Coupez)

Everything is in The Mesh (Thierry Coupez)



The Math for the All Fluids Approach

Regularity

$$u \in C^0(H^1_0 \cap W^{1,\infty}),\ \partial_t\phi + u\nabla\phi = 0 \Rightarrow \phi \in C^0(L^2)$$

If $\phi \in L^4(W^{1,4})$ then (u,p) exists in $L^2(H^1_0) \cap C^0(L^2)\ \times\ L^2(L^2)$ and

$$\rho_\phi(\partial_t u + u\cdot\nabla u) - \nabla\cdot(2\mu D(u) - pI) = \rho_\phi\vec{g},\ \ \nabla\cdot u = 0$$

- **No existence proof** (except $T << 1$) for the coupled problem

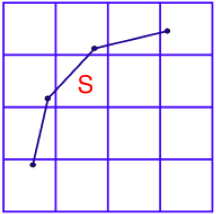
$$\rho = \rho^f \mathbf{1}_{x:\phi(x)<-\epsilon} + \rho^s \mathbf{1}_{x:\phi(x)>\epsilon} + \dots$$

- Marrouf-Bernardi show convergence of a Characteristic-Galerkin scheme + P^2/P^1 with error $O_\epsilon(h)$ and CFL $\delta t < C_\epsilon h$ if

$$\phi \in C^0(W^{2,\infty}),\ u \in W^{1,\infty}(]0,T[\times\Omega) \cap H^2(L^2) \cap C^0(H^2),\ p \in L^\infty(H^2)$$

Immersed Boundary Method (Charles Peskin)

$$-\Delta u = 0 \text{ in } \Omega \subset D, \quad u|_S = g$$



The Lagrange multiplier approach: find u, λ:

$$\int_D \nabla u \nabla v + \int_S \lambda v - \int_S (u-g)\mu = 0 \ \ \forall u, \mu$$

-inf-Sup condition needed for discrete space $V_h \times M_H \ \Rightarrow \ H_S > Ch_\Omega, \ C > 1$, Loose $\sqrt{h}$?, Girault &al[1999], Boffi-Giraldi &al[2003,2011,2015]

Extension to deformable bodies

As in Peskin's immersed boundary method : add surface forces F to model the reaction of the solid S on the fluid, and solves in a fixed domain with a non-body fitted mesh.

Immersed Boundary Method (II)

- New Result by Lucia Gastaldi, Daniele Boffi & Nicola Cavallini!
- Solid $\mathcal{B}$ volume/surface/curve in a fluid Ω. $\delta\rho = \rho^s - \rho^f$.

Let $X(s,t)$ be the position at t of a point s at $t = 0$ in the solid.

$$\rho\frac{d}{dt}(\mathbf{u}(t),\mathbf{v}) + a(\mathbf{u}(t),\mathbf{v}) + b(\mathbf{u}(t),\mathbf{u}(t),\mathbf{v})$$

$$-(\operatorname{div}\mathbf{v}, p(t)) + \mathbf{c}(\boldsymbol{\lambda}, \mathbf{v}(\mathbf{X}(\cdot,t))) = 0 \qquad \forall \mathbf{v} \in H_0^1(\Omega)^d$$

$$(\operatorname{div}\mathbf{u}(t), q) = 0 \qquad \forall q \in L_0^2(\Omega)$$

$$\delta\rho\int_{\mathcal{B}}\frac{\partial^2\mathbf{X}}{\partial t^2}\mathbf{Y}\,ds + \kappa\int_{\mathcal{B}}\nabla_s\mathbf{X}\,\nabla_s\mathbf{Y}\,ds$$

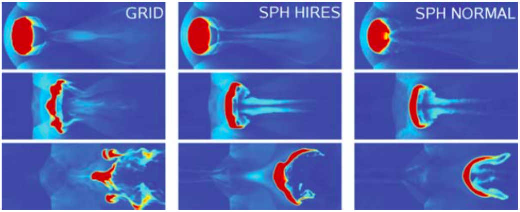
$$-\mathbf{c}(\boldsymbol{\lambda},\mathbf{Y}) = 0 \qquad \forall \mathbf{Y} \in H^1(\mathcal{B})^d$$

$$\mathbf{c}\left(\boldsymbol{\mu}, \mathbf{u}(\mathbf{X}(\cdot,t),t) - \frac{\partial\mathbf{X}(t)}{\partial t}\right) = 0 \qquad \forall \boldsymbol{\mu} \in \Lambda$$

- Existence, stability, convergence, stationary error estimate if $h_{\mathcal{B}} > Ch_\Omega$
- Solid is sum of fluid + elastic. Regularity of X? arxiv.org/abs/1407.5184

A Word on SPH

- Smooth Particle Hydrodynamics works for piecewise constant density flows
- Very popular in astrophysics/cosmology (code Zeus2)
- Shun by mathematicians as nonsense near boundaries ⇒ avoid them!



OSCAR AGERTZ ET AL, Mon. Not. R. Astron. Soc. 380, 963-978 (2007)

- New proof of convergence without boundaries but varying densities in JOEP H.M. EVERS, IASON A. ZISIS, BAS J. VAN DER LINDEN, MANH HONG DUONG, arXiv:1501.04512v1
- Requires some regularity in the distribution of particles

ALE Fluid + 3D-Structure

Fluid: Eulerian velocity. Solid Elasticity with Lagrangian small displacements.

- ALE Navier-Stokes for the fluid (A. Quarteroni et al)

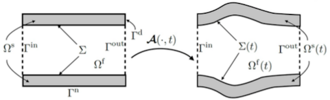
$$\rho^f\left(\partial_t|_{\mathcal{A}} u + (u-w)\cdot\nabla u\right) - \nabla\cdot\sigma^f(u,p) = 0, \quad \nabla\cdot u = 0$$

- Elasticity for the solid

$$\rho^s\partial_{tt}d - \nabla\cdot\Pi(d) = 0 \text{ in } \Omega^s$$
$$\Pi(d) = (I+\nabla d)(\lambda \mathrm{tr}E + 2\mu E), \quad E = \nabla d + \nabla d^T + \nabla d \nabla d^T$$

- Fluid-Solid Matching at Σ

$$u = w = \partial_t d, \quad \Pi(d)n^s = -\sigma^{\mathcal{A}}_{\mathbf{n}} := -\det\nabla\mathcal{A}^{-1}\sigma(u,p)\nabla\mathcal{A}^{-T}n^f \quad \text{on } \Sigma$$



Theorem(Formaggia-Moura-Nobile (2007)) Energy decays with viscosity

- Existence is not proved : Elasticity PDE does not give $\partial_t d \in H^{\frac{1}{2}}(\Sigma)$.

ALE Fluid + thin-Shell Structure

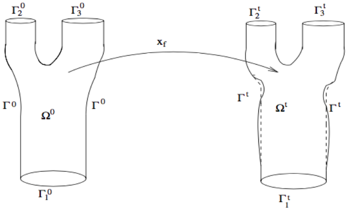
- ALE Navier-Stokes for the fluid

$$\rho^f\left(\partial_t|_{\mathcal{A}}u+(u-w)\cdot\nabla u\right)-\nabla\cdot\sigma^f(u,p)=0,\quad \nabla\cdot u=0 \quad \text{in } \Omega^f(t)$$

- Visco-Elasticity Koiter shell model reduced to normal displacement

$$\rho^s h^s\partial_{tt}\eta-\nabla\cdot(T\nabla\eta)-\nabla\cdot(C\nabla\partial_t\eta)+c\partial_t\eta+b\eta=-\sigma^{\mathcal{A}}_{\mathbf{nn}} \quad \text{on } \Sigma^s$$

- Matching velocities $u_n=w_n=\partial_t\eta$ on Σ



Fluid + Thin-Shell : Existence and Regularity

- Everything fits into a single variational formulation : $u = n\partial_t\eta$ and $\forall\xi, q, v : v\times n = 0$

$$\int_\Omega \Big[\rho^f(\partial_t|_{\mathcal{A}}u + (u-w)\cdot\nabla u)\cdot v + \frac{\mu}{2}(\nabla u + \nabla u^T):(\nabla v + \nabla v^T)$$
$$-p\nabla\cdot v + q\nabla\cdot u\Big] + \int_\Sigma [\rho^s h^s\partial_{tt}\eta\,\xi + \nabla\xi\cdot T\nabla\eta + b\eta\xi] = 0$$

Theorems

- Energy decays with viscosity (Nobile-Vergara (SIAM 2008))
- Solution exists (Chambolle-Esteban-Grandmont)

$$\rho^s h^s\partial_{tt}\eta + \Delta^2\eta + \epsilon\Delta^2\partial_t\eta - \nabla\cdot(T\nabla\eta) - \nabla\cdot(C\nabla\partial_t\eta) + c\partial_t\eta + b\eta = -\sigma^f_{nn} \text{ on } \Sigma^s$$

- Muha-Canic : Alternative proof via a Algorithmic decomposition of operator
- Grandmont: $\epsilon \to 0$ OK

Algorithms

- Fluid-Structure in one formulation: Most natural to solve all at once:Safe but expensive!
- Separate formulations + maching :Build the matrices and solve the full system at the matrix level by clever splitting (Idelsohn-Oñate)
- Fluid+structure+matching Most natural to iterate at each time step: Fluid then structure then update the geometry:
  - Very slow (especially if $\rho^f \sim \rho^s$ by added mass effect)
  - Large literature on speed up and stability conditions by operator splitting (Canic et al), Algebraic factorization (Badia-Quaini), Robin conditions for the matching (M. Fernandez) etc.
  - OK if $\delta t = O(h)$, some are unconditionally stable when the domain motion is neglected.

Can the Motion of the Domain be Neglected?

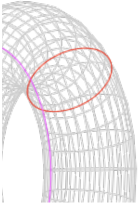
Transpiration Condition

Σ_t the moving boundary, Σ a reference bdy: $\Sigma_t = \{x + \eta\vec{n} : x \in \Sigma\}$

$$u(x + \eta\vec{n}) = \vec{n}\partial_t\eta(x),\ x \in \Sigma \ \Rightarrow\ u + \eta\frac{\partial u}{\partial n} = \vec{n}\partial_t\eta + o(\eta) \text{ on } \Sigma$$

Let torus(r, R) be tangent to Σ

$$\nabla \cdot u = 0 \Rightarrow n \cdot \frac{\partial u}{\partial n} = (1 + \frac{r}{R}\cos^2\theta)\frac{u \cdot n}{r} \Rightarrow$$
$$u(x + \eta\vec{n}) \cdot n = u \cdot n\Big(1 + \frac{\eta}{r}(1 + \frac{r}{R}\cos^2\theta)\Big)$$



Similarly $\sigma^f_{nn} = p + 2(1 + \frac{r}{R}\cos^2\theta)\frac{\mu}{r}u \cdot n$

Hence

$$u \cdot n = \partial_t\eta/\Big(1+\frac{\eta}{r}(1 + \frac{r}{R})\Big), p \approx b\eta + \partial_{tt}\eta - \nabla \cdot (T\nabla\eta) + \frac{2\mu\partial_t\eta}{r}(1 + \frac{r}{R} - \frac{\eta}{r})$$

Fluid + Shell + Computation on a Fixed domain

- Find $\eta, p, u : u = n\partial_t\eta$ and $\forall \xi, q, v : v \times n = 0$ and

$$\int_\Omega \left[\rho^f(\partial_t u - u \times \nabla \times u) \cdot v + \frac{\mu}{2}(\nabla u + \nabla u^T) : (\nabla v + \nabla v^T)\right.$$
$$\left. - p\nabla \cdot v + q\nabla \cdot u\right] + \int_\Sigma \left[\rho^s h^s \partial_{tt}\eta\, \xi + \nabla\xi \cdot T\nabla\eta + b\eta\xi\right] = 0$$

- Simulate the wall motion with η
- Simplifications: differentiate in t the shell equation and use $\partial_t\eta = u_n$
- Find u, p such that $u \times n = 0$ and for all v with $v \times n = 0$ and $\forall q$

$$\int_\Omega \left[\rho^f(\partial_t u - u \times \nabla \times u) \cdot v + \frac{\mu}{2}(\nabla u + \nabla u^T) : (\nabla v + \nabla v^T)\right.$$
$$\left. - p\nabla \cdot v + q\nabla \cdot u\right] + \int_{\Sigma\times(0,t)} \left[\rho^s h^s \partial_{tt} u_n\, v_n + T\nabla u_n \cdot \nabla v_n + b u_n\, v_n\right] = 0$$

The final model

Better apply elimination of η on the time discretised original equations.
Variational formulation (after division by ρ^f: $\forall \hat{u} \in H^1(\Omega)^3,\ \hat{p} \in L^2(\Omega)$,

$$\int_\Omega \Big[\hat{u}\cdot(\frac{u^{m+1}-u^m}{\delta t} - u^{m+\frac12}\times\nabla\times u^{m+\frac12}) - p^{m+1}\nabla\cdot\hat{u} - \hat{p}\nabla\cdot u^{m+\frac12}$$
$$+\nu\nabla\times u^{m+\frac12}\cdot\nabla\times\hat{u}\Big] + \int_{\partial\Omega}[\frac{1}{\epsilon}u^{m+\frac12}\times n\cdot\hat{u}\times n + b\hat{u}\cdot(u^{m+\frac12}\delta t + U^m)]$$
$$+\int_{\partial\Omega}[\partial_{tt}u|^{m+\frac12}\cdot\hat{u} + \nabla u^{m+\frac12}\mathbf{T}\nabla\hat{u}] = 0,\ U^{m+1} = U^m + u^{m+\frac12}\delta t$$

Energy is preserved

$$\hat{u} = u^{m+\frac12}, \hat{p} = -p^{m+1} \Rightarrow \|u^{m+1}\|_0^2 + \nu\delta t\sum_{k\le m}\left(\|\nabla u^{k+\frac12}\|_0^2 + b\delta t^2\int_{\partial\Omega}|u^{k+\frac12}|^2\right)$$
$$+\frac{1}{2b}\int_{\partial\Omega}[\sum_{k\le m}(p^{k+1}-p^k)^2 - {p^{m+1}}^2 + {p^0}^2] = \|u^0\|_0^2$$

RUN

The final model

Better apply elimination of η on the time discretised original equations.
Variational formulation (after division by ρ^f: $\forall \hat{u} \in H^1(\Omega)^3,\ \hat{p} \in L^2(\Omega)$,

$$\int_\Omega \Big[\hat{u}\cdot\Big(\frac{u^{m+1}-u^m}{\delta t} - u^{m+\frac12}\times\nabla\times u^{m+\frac12}\Big) - p^{m+1}\nabla\cdot\hat{u} - \hat{p}\nabla\cdot u^{m+\frac12}$$
$$+\nu\nabla\times u^{m+\frac12}\cdot\nabla\times\hat{u}\Big] + \int_{\partial\Omega}\Big[\frac{1}{\epsilon}u^{m+\frac12}\times n\cdot\hat{u}\times n + b\hat{u}\cdot(u^{m+\frac12}\delta t + U^m)\Big]$$
$$+\int_{\partial\Omega}[\partial_{tt}u|^{m+\frac12}\cdot\hat{u} + \nabla u^{m+\frac12}\mathbf{T}\nabla\hat{u}] = 0,\ \ U^{m+1} = U^m + u^{m+\frac12}\delta t$$

Energy is preserved

$$\hat{u} = u^{m+\frac12}, \hat{p} = -p^{m+1} \Rightarrow \|u^{m+1}\|_0^2 + \nu\delta t\sum_{k\le m}\Big(\|\nabla u^{k+\frac12}\|_0^2 + b\delta t^2\int_{\partial\Omega}|u^{k+\frac12}|^2\Big)$$
$$+\frac{1}{2b}\int_{\partial\Omega}\Big[\sum_{k\le m}(p^{k+1}-p^k)^2 - {p^{m+1}}^2 + {p^0}^2\Big] = \|u^0\|_0^2$$

RUN

Validation

- Well Posedness
- Convergence of the time scheme
- Convergence of the time-space scheme
- Comparison with other models
- Numerical performance

Remark

• It is OK to use $(\nabla\times u, \nabla\times\hat{u})$ instead of $(\nabla u+\nabla u^T \ : \ \nabla\hat{u}+\nabla\hat{u}^T)$ because they are equal. Except for corner singularities!

• The use of $-u\times\nabla\times u$ instead of $u\cdot\nabla u-\frac{1}{2}\nabla|u|^2$ is within the small displacement approximation.

Convergence (with T. Chacon, V. Girault, F. Murat)

Lemma If Ω is $\mathcal{C}^{1,1}$ or polyhedric and $u_0 \in L^2(\Omega)^3$, $p_0 \in H^{1/2}(\Sigma)$, then the weak solution of the continuous problem verifies $u \in L^2(\mathbf{H}^2)$, $\partial_t u \in L^2(\mathbf{L}^2)$, $p \in L^2(H^1)$, and $u \times n = 0$ in $L^2(L^4(\Sigma))$, $\partial_t p = bu \cdot n$ in $L^2(H^{1/2}(\Sigma))$, $p(0) = p_0$

Theorem The solution of the time discretized variational problem satisfies

$$\|u_\delta\|_{L^\infty(\mathbf{L}^2)} + \sqrt{\nu}\,\|u_\delta\|_{L^2(\mathbf{H}^1)} + \quad b\,\|\delta t \sum_{k=1}^{n+1} u^k \cdot n\|_{L^\infty(\mathbf{L}^2(\Sigma))}$$
$$\leq C\left(\|u_0\|_{0,2,\Omega} + \frac{1}{\sqrt{\nu}}\|p_0\|_{L^2(\Sigma)}\right)$$

where u_δ is the time-linear interpolation of $\{u^n\}_0^N$.

Theorem If Ω is simply connected, $\exists$ subsequence $(u_{\delta'}, p_{\delta'})$ which converges to the continuous problem in $L^2(\mathbf{W}) \times H^{-1}(L^2)$ where

$$\mathbf{W} \;=\; \{w \in L^2(\Omega) \,|\, \nabla \times w \in L^2(\Omega),\ \nabla \cdot w \in L^2(\Omega),\ n \times w_{|_\Sigma} = \mathbf{0}\}.$$

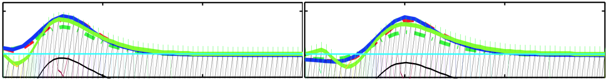
Proof uses the continuous embedding of $\mathbf{W}_0$ in $\mathbf{H}^1_0$.

Comparison with Bukača et al.[1]

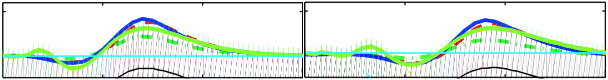
- Flow between two // compliant walls $(0, L) \times (-R, R)$ with 60×10 grid.

$$L = 6,\ R = 0.5,\ p = 10^4(1 - \cos(\frac{2\pi\ t}{t_m}))\ \text{if}\ t < t_m,\ \text{else}\ 0;\ \ t_m = 5\ 10^{-3}.$$

- $\delta t = 10^{-4},\ \nu = 0.035,\ \tilde{b} = 4\ 10^5,\ \tilde{\mathbf{T}} = 2.5\ 10^4,\ \ \bar{h} = 0.1,\ \ \frac{\rho^s}{\rho^f} = 1.1.$



t=6ms t=8ms

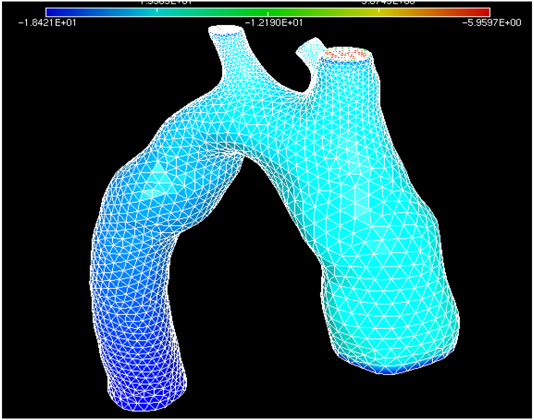


t=10ms t=12ms

Zoom near the compliant wall (in green) and Bukaca et al (dark blue) + other's results .

[1] M. BUKAČA, S. ČANIČ, R. GLOWINSKI, J. TAMBACAC, A. QUAINI. FSI in blood flow capturing non-zero longitudinal structure displacement. JCP 235 (2013) 515-541

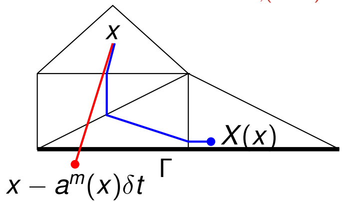
Simulation of an aortic bend



Note: Better $u \cdot \nabla u - \frac{1}{2}\nabla|u|^2$ and Characteristic - Galerkin approximation

Speed-up with Galerkin - Characteristic Method (I)

Don't upwind or if you do, use this:



$$\partial_t u + a \cdot \nabla u|_{x,(m+1)\delta t} = \frac{u^{m+1}(x) - u^m(x - a^m(x)\delta t)}{\delta t} + O(\delta t)$$
$$= \frac{u^{m+1} - u^m oX}{\delta t} + O(\delta t)$$

with $X(x) = \mathcal{X}_{a^m}(m\delta t)$ and

$$\frac{d\mathcal{X}}{d\tau}(\tau) = a^m(\mathcal{X}(\tau)), \quad \mathcal{X}((m+1)\delta t) = x$$

Second order approximation

$$\partial_t u + a \cdot \nabla u|_{x,(m+1)\delta t} \approx \frac{3u^{m+1}(x) - 4u^m(x - a^m(x)\delta t) + u^{m-1}((x - 2a^m(x)\delta t))}{2\delta t}$$
$$= \frac{3u^{m+1} - 4u^m oX^*_{\delta t} + u^{m-1} oX^*_{2\delta t}}{2\delta t} + O(\delta t^2)$$

with $X^*_{k\delta t}(x) = \mathcal{X}_{a^{*\ m+\frac{1}{2}}}(k\ m\delta t), \ k = 1, 2$

and $a^{*\ m+\frac{1}{2}} = 2a^m - a^{m-1}$

Galerkin - Characteristic Method (II)

Zhiyong Si's modified artificial viscosity[1]

$$\partial_t u + a \cdot \nabla u - \nu \Delta u = 0, \quad u(0), \quad u|_\Gamma \text{ given}$$

- Step 1 $$\frac{3u^{m+\frac{1}{2}} - 4u^m o X^*_{\delta t} + u^{m-1} o X^*_{2\delta t}}{2\delta t} - (\nu + \sigma h)\Delta u^{m+\frac{1}{2}} = 0$$
- Step 2 $$\frac{3u^{m+1} - 4u^m o X^*_{\delta t} + u^{m-1} o X^*_{2\delta t}}{2\delta t} - (\nu + \sigma h)\Delta u^{m+1} + \sigma h \Delta u^{m+\frac{1}{2}} = 0$$

Theorem After discretization with a finite element method of order k,

$$\|u^{m+1} - u_h^{m+1}\|_0 \le C(\delta t^2 + h^{k+1} + \sigma^2 h^2 + \delta t \sigma h)$$

$$\left(\nu \delta t \sum_{j \le m} \|u^{m+1} - u_h^{m+1}\|_0^2\right)^{\frac{1}{2}} \le C(\delta t^2 + h^k + \sigma^2 h^2 + \delta t \sigma h)$$

And for N.S. $$\left(\delta t \sum_{j \le m} \|p^m - p_h^m\|\right)^{\frac{1}{2}} \le C(\delta t^2 + h^k + h^2 + \delta t^2 h).$$

[1] ZHIYONG SI. Second order modified method of characteristics mixed defect-correction finite element method for time dependent Navier-Stokes problems. Numer Algor (2012) 59:271-300.

Galerkin-Characteristics (III)

Estimates are destroyed by quadrature error $I = \int_\Omega u_h^m(X(x))w_h(x)\mathrm{d}x$. Only estimate known is for quadrature at 3 vertices q_i^j of triangle T_j

$$I \approx \sum_j \sum_{i=1,2,3} u_h^m(X(q_i^j))w_h(q_i^j)\frac{|T_j|}{3} \Rightarrow \|u - u_h\|_\infty \le C_\epsilon(h + \delta t + \frac{h^{2-\epsilon}}{\delta t})$$

In practice a Gauss quad of degree 5 works fine. Correction to be exactly conservative c by J. Rappaz, (also tested with freefem++).

In the end one solve at each time step a generalized Stokes problem independent of time.
RUN
Runs also in parallel with MPI

[1] OP and M.Tabata. Stability and convergence of a Galerkin-characteristics finite element scheme of lumped mass type Int. J. Numer. Meth. Fluids 2010; 64:1240-1253

[2] J. Rappaz, S. Flotron Numerical conservation schemes for convection-diffusion equations (to appear)

Conclusions on the Model

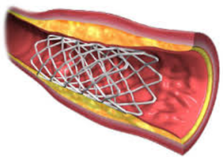
- The model seems to be precise within the small disturbance approximation,
- It is unconditionally stable,
- It is fast
- Suited to inverse problems $\Rightarrow$

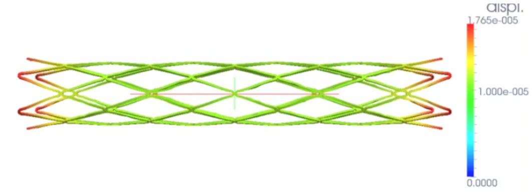
The group REO at INRIA [1] has done several studies on the recovery of parameters of the model using non-linear Kalman filters

[1] C. Bertoglio, D. Barber, N. Gaddum, I.Valverde, M. Rutten, P. Beerbaum, P. Moireau, R. Hose, J-F. Gerbeau Identification of artery wall stiffness: In vitro validation and in vivo results of a data assimilation procedure applied to a 3D fluid-structure interaction model. J. Biomechanics 47 (2014) 1027-1034

Optimal Stent with the Surface Pressure Model

A stent tuned to the patient? e.g. $\min_{b(x)} J = \int_{\Sigma\times(0,T)} F(p)dxdt,\ F(p) = \frac{1}{s}(p-p_d)^s$





S. Čaniċ[1] (with freefem++)

Subject to

$$\int_\Omega [\hat{u}\cdot(\frac{u^{m+1}-u^m}{\delta t} - u^{m+1}\times\nabla\times u^m) - p^{m+1}\nabla\cdot\hat{u} - \hat{p}\nabla\cdot u^{m+1}]$$
$$+\int_\Omega \nu\nabla\times u^{m+1}\cdot\nabla\times\hat{u} + \int_\Sigma r(U^{m+1}-U^m-u^{m+1}\delta t)$$
$$+\int_\Sigma b(u^{m+1}\delta t + U^m)\cdot\hat{u}^{m+1} = 0\ \ \forall r,\quad \hat{u}\in V_h, \hat{p}\in Q_h:\ \hat{u}\times n|_\Gamma = 0$$

[1] J. Tambaca, S. Čaniċ, M. Kosor, R.D. Fish, D. Paniagua. Mechanical Behavior of Fully Expanded Commercially Available Endovascular Coronary Stents. Tex Heart Inst J 2011;38(5):491-501).
[2]J. Tambaca, M. Kosor, S. Čaniċ, and D. Paniagua. Mathematical Modeling of Endovascular Stents. SIAM J Appl Math. Volume 70, Issue 6, pp. 1922-1952 (2010)
[3]S. Čaniċ, J. Tambaca. "Cardiovascular Stents as PDE Nets: 1D vs. 3D." IMA J. Appl. Math. 77(6): pp 748-770, 2012.

Stent Mesh with `freefem++`(F. Hecht)

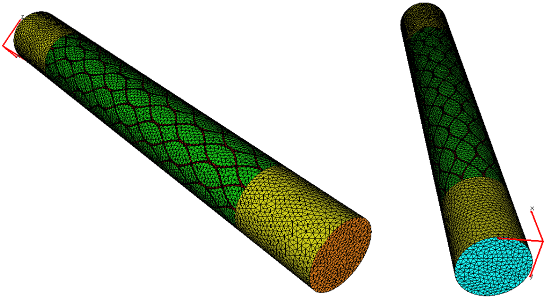


Figure: Regions: in (x=0,blue), out (x=L,orange), stent (red), cylinder off stent (green), buffer before and after (yellow). Dimensions $R = 1$, $L = C_l(N_L + N_R + N_{LL})$ where the length in axial direction of the cell is $C_l = \frac{2\pi R L_{cp}}{N_R H_{cp}}$ with $N_L = 8$ cells in axial direction, $N_R = 10$ vells in radial direction, $N_{LL} = 2$ cells before and also after the stent (buffer). $L_{cp} = 2$ (resp $H_{cp} = 2$ is the width (resp height) of the stent cell.

Stent Experiments (I)

10 time steps per iteration cycle, 1 iteration CPU = 27min nb of elements 286110, nb of vertices 51448 nb of edges = 26538 nb of Nodes 337558 nb of DoF 1064122

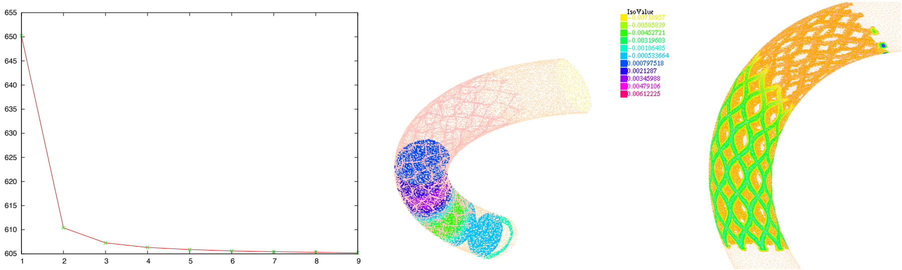
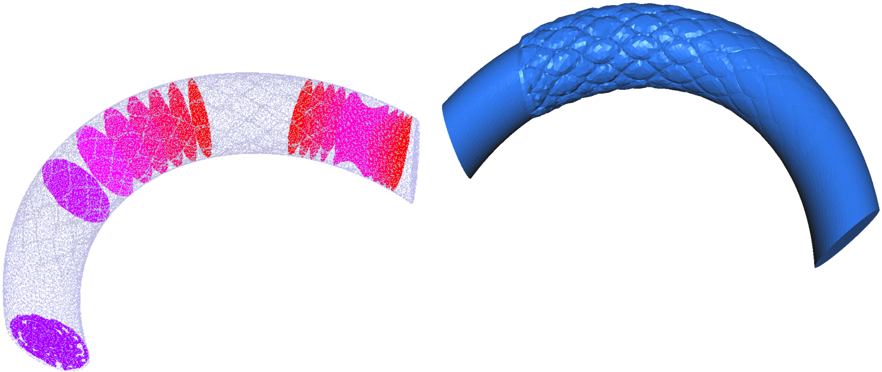


Figure: *9 iterations of optimisation. Gradient. Value of b*

Stent Experiments (II)



Pressure Geometry deformation

Conclusion and Perspectives

1. A lot is happening in applied math for FSI problems
2. Multi-fluid penalty with advanced mesh generator
3. Immersed boundary method now has a solid base
4. ALE may not be the best idea
5. `freefem++` is useful for hemodynamics to prototype new ideas.

Many things to do:

1. Large displacement, contact
2. More inverse problems
3. Validate a Chorin-Rannacher decomposition

Thanks for the Invitation