Thin Structure of Eigenvalue Clusters for Non-Hermitian Toeplitz Matrices

E. E. Tyrtyshnikov and N. L. Zamarashkin [1]

Institute of Numerical Mathematics,
Russian Academy of Sciences,
Gubkina 8, Moscow 117333, Russia

ABSTRACT

In contrast to the Hermitian case, the "unfair behavior" of non-Hermitian Toeplitz eigenvalues is still to be unravelled. We propose a general technique for this, which reveals the eigenvalue clusters for symbols from L_∞. Moreover, we study a thin structure of those clusters in the terms of properly defined subclusters. In some cases, this leads to as much as the Szego-like formulas.

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1 Introduction

At present, we know a lot about the asymptotic behavior of Toeplitz eigenvalues for real-valued symbols, even for multilevel Toeplitz matrices [7, 13, 14]. For complex-valued and even block-valued symbols, we are aware how the singular values behave [1, 8, 10, 13, 14]. In all these cases, we enjoy the Szego-like formulas. To keep the same formula for general Toeplitz non-Hermitian eigenvalues, we need to apply it only with harmonic test functions [10], which skip too many details. As a matter of fact, the results for general Toeplitz non-Hermitian eigenvalues are rather thin on the ground. What we are aware of is based on profound and intricate studies of the Fisher–Hartwig hypothesis [2, 3, 16]. In this paper we propose a different and far simpler approach. In effect it is more general as it may apply to not necessarily Toeplitz matrices.

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Given a sequence of matrices $A_n \in \mathbb{C}^{n\times n}$, consider a set Ω on the complex plane and let $\gamma(A_n, \Omega, \varepsilon)$ count how many eigenvalues of A_n fall outside the ε-extension of Ω. We say that Ω is a *subcluster* for the eigenvalues of A_n if

$$c(\Omega, \varepsilon) \equiv \limsup \frac{\gamma(A_n, \Omega, \varepsilon)}{n} < 1,$$

for any $\varepsilon > 0$. If $c(\Omega, \varepsilon) = 0$ for any $\varepsilon > 0$, then Ω is called a *cluster* [13]. In the latter case we write $\lambda(A_n) \sim \Omega$.

Denote by $\overline{\Omega}$ the topological closure of Ω. If for any $z \notin \overline{\Omega}$ there is a bounded open simply connected domain $\Omega' \supset \overline{\Omega}$ with a closed analytical boundary passing through z, then Ω will be called a *fair domain*. (It is equivalent to say that $\overline{\Omega}$ is a compact set with connected complement.)

Our approach can be sketched as follows. Since it might be difficult to say anything about $\lambda(A_n)$ directly, we may try to approximate A_n by some simpler matrices B_n with a cluster Ω. We only need to know when it follows that Ω is also a cluster for $\lambda(A_n)$. In many cases, it is sufficient to know that $||A_n - B_n||_F^2 = o(n)$ [13, 15]. It is so when A_n and B_n are Hermitian. However, in this paper we discover how the same key relation may work for non-Hermitian A_n and B_n. More precisely, we prove the following.

Theorem 1.1 *Given two sequences of matrices A_n and B_n of order n, assume that*

(1) $||A_n - B_n||_F^2 = o(n)$;

(2) A_n *and* B_n *are bounded in the spectral norm uniformly in* n;

(3) B_n *are normal matrices;*

(4) $\lambda(B_n) \sim \Omega$, *where* Ω *is a union of* m *fair domains* Ω_k, $1 \le k \le m$, *with pairwise disjoint closures.*

Then

(a) $\lambda(A_n) \sim \Omega$, *and, moreover, for all sufficiently small* $\varepsilon > 0$,

(b) $\gamma(A_n, \Omega_k, \varepsilon) - \gamma(B_n, \Omega_k, \varepsilon) = o(n)$, $1 \le k \le m$.

Formally, (a) follows from (b). However, we prefer to state (a) explicitly and first. Above all, it is due to our way to this theorem. We prove first that Ω is a common cluster for A_n and B_n. Only having this done, we get on to a thin structure of this cluster. As (b) reveals, the concerned thin structure of Ω is also common for A_n and B_n. Consequently, if Ω_k is a subcluster for $\lambda(B_n)$, then it is also a subcluster for $\lambda(A_n)$.

Now, consider Toeplitz matrices $A_n = A_n(f) = [a_{i-j}]$ with the entries from the Fourier expansion

$$f(x) \sim \sum_{k=-\infty}^{\infty} a_k \, e^{\mathrm{i}kx};$$

f is referred to as a symbol (generating function) for A_n. We may try as B_n the optimal circulants $C_n(f)$ (those that minimize $||A_n - C_n||_F$ over all circulants C_n [5]). Then, the hypotheses (1)–(3) of Theorem 1.1 are fulfilled as soon as $f \in L_\infty$. Since the eigenvalues of $C_n(f)$ are distributed as the values of $f(x)$ (it is proved in [13] even for $f \in L_1$), we come up with the following theorem.

Theorem 1.2 *Let $f \in L_\infty$ be complex-valued, and assume that the values $f(x)$ almost everywhere in the Lebesgue sense are located inside Ω, a union of m fair domains Ω_k, $1 \le k \le m$, with pairwise disjoint closures. Then Ω is a cluster for $\lambda(A_n(f))$, and, moreover, for all sufficiently small $\varepsilon > 0$,*

$$\lim_{n\to\infty} \frac{\gamma(A_n, \Omega_k, \varepsilon)}{n} = 1 - c_k, \quad 1 \le k \le m,$$

where

$$c_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} \chi_k(f(x))dx, \quad \chi_k(z) = \begin{cases} 1, & z \in \Omega_k \\ 0, & z \notin \Omega_k. \end{cases}$$

The paper is organized as follows. In Section 2, we gather some auxiliary statements. In Section 3, we present the proof of Theorem 1.1. We start with the claim (a). The main tools are special polynomial mappings for prescribed nested domains to march inside some nested disks. Then, for the assertion (b), we use essentially the already established (a) while the domains are shifted to inside some ε-tubes around the real axis.

In Section 4, we get to the case of non-Hermitian Toeplitz matrices and expound the proof of Theorem 1.2. In Section 5, we discuss a more detailed structure of clusters. Then, we finish with a special case when the values $f(x)$ lie on open curves. For such cases we propose the Szego-like formula.

2 Auxiliary statements

Lemma 2.1 *Given two sequences of $n \times n$ matrices A_n and B_n, assume that*

(1) $||A_n - B_n||_F^2 = o(n)$, *and*

(2) $||B_n||_2 \le M$ *uniformly in n.*

Then $\lambda(A_n) \sim K(M) \equiv \{z : |z| \le M\}$.

Proof. Consider the Schur decomposition $A_n = U_n T_n U_n^*$ with an upper triangular T_n, and, using the same unitary U_n, write $B_n = U_n \tilde{B}_n U_n^*$. Then,

$$\sum_{i=1}^{n} |t_{ii} - \tilde{b}_{ii}|^2 \le ||T_n - \tilde{B}_n||_F^2 = ||A_n - B_n||^2 = o(n).$$

Let $\gamma_n(\varepsilon)$ denote the number of those t_{ii} that escape from $K(M+\varepsilon)$. Since $|\tilde{b}_{ii}| \le M$, we obtain $\gamma_n(\varepsilon)\, \varepsilon^2 = o(n)$, and, hence, $\gamma_n(\varepsilon) = o(n)$. □

Lemma 2.2 *Let $A, H \in \mathbb{C}^{n\times n}$ and $H = H^*$. Then the eigenvalues of A and H can be ordered so that*

$$\sum_{i=1}^{n} |\lambda_i(A) - \lambda_i(H)|^2 \ \le\ 2\, ||A - H||_F^2.$$

Proof. Using the Schur decomposition for A, we reduce the problem to the same one with an upper triangular A. Let $A = \Lambda + U$, where Λ is diagonal and U is strictly upper triangular. Since Λ and H are normal matrices, we may apply the Hoffman–Wielandt theorem. Let $H = D + R + R^*$, where D is diagonal and R is strictly upper triangular. Then,

$$||\Lambda - H||_F^2 = \left(||\Lambda - D||_F^2 + \ ||R||_F^2\right) + \left(||R||_F^2\right) \ \le\ 2\, ||A - H||_F^2,$$

because each term of the two does not exceed $||A - H||_F^2$. □

Remark. Lemma 2.2 can be found in [9] with a precise specification of the eigenvalue ordering. It is sufficient to order them so that

$$\mathrm{re}(\lambda_1(A)) \geq \ldots \geq \mathrm{re}(\lambda_n(A)) \quad \text{and} \quad \lambda_1(H) \geq \ldots \geq \lambda_n(H).$$

Lemma 2.3 *Given two sequences of $n \times n$ matrices A_n and B_n, assume that*

(1) $||A_n - B_n||_F^2 = o(n)$;

(2) $\left\| \frac{B_n - B_n^*}{2} \right\|_F^2 \leq \varepsilon\, n$, *for some $\varepsilon > 0$ and for all sufficiently large n.*

Then the eigenvalues of A_n and B_n can be indexed so that

$$\sum_{i=1}^{n} |\lambda_i(A_n) - \lambda_i(B_n)|^2 \;\leq\; c\, \varepsilon\, n$$

for all sufficiently large n, with c an absolute constant.

Proof. Let $H_n = (B_n + B_n^*)/2$. Then $||A_n - H_n||_F^2 \leq o(n) + \varepsilon\, n$ and also $||H_n - B_n||_F^2 \leq \varepsilon\, n$. It remains to apply the previous lemma twice. □

3 General approach

We present here the proof of Theorem 1.1. However, this is more than merely a proof, for we give a general technique for clusters on the complex plane.

1. To brush up the idea of coming to (a), assume that $\Omega = \Omega_1$ is a disk centered at zero. Consider a larger disk Ω' of radius M. Since $\lambda(B_n) \sim \Omega$, there can be only $o(n)$ eigenvalues outside Ω'. Taking into account that B_n are normal, consider the spectral decomposition of B_n and, keeping the same eigenvector matrices, modify the large eigenvalues of B_n to obtain some B_n' with the eigenvalues inside Ω'. Since B_n' are normal, $||B_n'||_2 \leq M$ and, from the uniform boundedness of B_n in the spectral norm, we obtain $||B_n - B_n'||_F^2 = o(n)$, and hence, $||A_n - B_n'||_F^2 = o(n)$. Now, by Lemma 2.1, $\lambda(A_n) \sim \Omega'$, and, eventually, $\lambda(A_n) \sim \Omega$.

2. The considered above case is a bit too special. However, if Ω' is a bounded open simply connected domain, then, due to Riemann, there is a conformal mapping ϕ such that $\phi(\Omega') = D(M)$ is an open disk of radius M. Due to Runge, on any compact set $K \subset \Omega'$, the analytical function $\phi(z)$ can be uniformly approximated by a polynomial $p(z)$. Taking up a smaller Ω', if necessary, we may suppose that $K = \overline{\Omega'}$ (the closure of Ω'). Thus, for any $\delta > 0$, there is a polynomial $p(z)$ such that $p(\Omega') \subset D(M+\delta)$. Assume that $\Omega = \Omega_1 \subset \Omega'$ and at most $o(n)$ eigenvalues of B_n may stray out of Ω'. These eigenvalues can be modified, as above, to get to some normal matrices B_n' such that $\lambda(B_n') \subset \Omega'$ and $||B_n - B_n'||_F^2 = o(n)$. It is easy to verify that

$$||p(A_n) - p(B_n')||_F^2 = o(n) \quad \text{and} \quad ||p(B_n')||_2 \leq M + \delta.$$

By Lemma 2.1, again, $D(M+\delta)$ is a cluster for the eigenvalues of $p(A_n)$. This proves also that if a set O is such that $\overline{p(O)}$ and $\overline{D(M+\delta)}$ do not intersect, then O contains at most $o(n)$ of the eigenvalues of A_n.

Still, we can not stop at this, because we need to prove the same for any O with no common points with an ε-extension of Ω, and note that, if p is fixed, some of those O might be such that $\overline{p(O)} \cap \overline{D(M+\delta)} \neq \emptyset$.

3. Let $z \notin \overline{\Omega}$. Then there is an open set $O(z)$ such that $z \in O(z)$ and, for some polynomial p and $M > 0$, it holds

$$\overline{p(\Omega)} \subset D(M) \quad \text{and} \quad \overline{p(O(z))} \cap \overline{D(M)} = \emptyset. \qquad (*)$$

To prove this, note that Ω is also a fair domain (as a union of finitely many fair domains with pairwise disjoint closures). Hence, we can take up a bounded open simply connected domain Ω'' such that $\overline{\Omega} \subset \Omega''$ and the boundary $\partial\Omega''$ is a closed analytical curve passing through z. Let ϕ be a conformal mapping such that $\phi(\Omega'') = D(M'')$ is a disk of radius M''. Since $\partial\Omega''$ is an analytical curve, we may extend ϕ through the boundary onto a larger bounded open simply connected domain, say Ω'''. Let $\phi(\Omega''') = D(M''')$ with $M''' > M''$. At the same time, there is a domain $\Omega' \supset \overline{\Omega}$ such that $\phi(\Omega') = D(M')$ for some $M' < M''$. Now, let $O(z)$ be the preimage of some sufficiently small open disk centered at $\phi(z)$. Using the Runge theorem, we approximate ϕ by a polynomial p. If p is sufficiently close to ϕ on Ω''', then we have $(*)$ for some $M' < M < M''$.

4. We are now ready to prove that $\lambda(A_n) \sim \Omega$. Consider a compact K disjoint with $\overline{\Omega}$ and yet nearby and large so that $\lambda(A_n) \subset K \cup \Omega_\varepsilon$ (Ω_ε is the ε-extension of Ω). By Article 3, K can be covered by open sets $O(z)$, $z \in K$, each having only $o(n)$ of the eigenvalues of A_n. Since K is a compact, there is a finite subcovering of K with some $O(z_1), \dots, O(z_N)$ that contain at most $o(n)$ of the eigenvalues of A_n. The proof of the assertion (a) of Theorem 1.1 is thus completed.

5. We get on to the claim (b) of Theorem 1.1. Consider any bounded open simply connected domains $\Omega'_k \supset \overline{\Omega_k}$ with closed Jordanian boundaries and pairwise disjoint closures. Denote by $D_k(\varepsilon')$ the disks of radius ε' centered at $w_k = k$ (it is important only that they are separated by a distance independent of ε'), and let ϕ be a mapping defined on the union Ω' of Ω'_k so that it maps conformally Ω'_k onto $D_k(\varepsilon')$, $1 \le k \le m$. Since the completion to Ω' is a connected set, by the extended Runge theorem, this analylitic on Ω' function ϕ can be uniformly approximated by a polynomial p on any compact $K \subset \Omega'$. We do not lose the generality assuming that $K = \overline{\Omega'}$. For some $\varepsilon > \varepsilon'$, we thus obtain $p(\Omega'_k) \subset D_k(\varepsilon)$ for all k. As previously, we can modify the eigenvalues of B_n to march them down to Ω'. The modified normal matrices B'_n still satisfy $||A_n - B'_n||_F^2 = o(n)$. It is easy to see that

$$||p(A_n) - p(B'_n)||_F^2 = o(n) \quad \text{and} \quad \left\| \frac{p(B'_n) - (p(B'_n))^*}{2} \right\|_2 \le \varepsilon.$$

6. By Lemma 2.3, the eigenvalues of $p(A_n)$ can match those of $p(B'_n)$ so that, for all sufficiently large n,

$$\sum_{i=1}^{n} |\lambda_i(p(A_n)) - \lambda_i(p(B'_n))|^2 \le c\, \varepsilon\, n.$$

Now, take up some $\zeta(\varepsilon) > 0$ and let γ_n count those indices i for which $|\lambda_i(p(A_n)) - \lambda_i(p(B'_n))| > \zeta(\varepsilon)$. From the above,

$$\gamma_n \ \le \ c\, \frac{\varepsilon}{\zeta^2(\varepsilon)}\, n.$$

To keep γ_n small for small ε, we can set $\zeta(\varepsilon) = \varepsilon^{1/3}$. Anyway, the choice of $\zeta(\varepsilon)$ should provide

$$\lim_{\varepsilon \to +0} \zeta(\varepsilon) = 0, \quad \lim_{\varepsilon \to +0} \frac{\varepsilon}{\zeta^2(\varepsilon)} = 0.$$

7. For any matrix A and a set D, let $\Gamma\,(A,\, D)$ designate the number of the eigenvalues of A belonging to D. From the previous article, with $\varepsilon' \equiv \varepsilon + \zeta(\varepsilon)$, we obtain

$$\Gamma\,(p(A_n),\, D_k(\varepsilon)) \;\le\; \Gamma\,(p(B'_n),\, D_k(\varepsilon')) + c\,\frac{\varepsilon}{\zeta^2(\varepsilon)}\,n,$$

and, similarly,

$$\Gamma\,(p(B'_n),\, D_k(\varepsilon)) \;\le\; \Gamma\,(p(A_n),\, D_k(\varepsilon')) + c\,\frac{\varepsilon}{\zeta^2(\varepsilon)}\,n.$$

By the construction of B'_n,

$$\Gamma\,(p(B'_n),\, D_k(\varepsilon')) \;=\; \Gamma\,(p(B'_n),\, D_k(\varepsilon)) \;=\; \Gamma\,(B'_n,\, \Omega'_k),$$

and, by the already established clustering property,

$$\Gamma\,(p(A_n),\, D_k(\varepsilon')) \;=\; \Gamma\,(p(A_n),\, D_k(\varepsilon)) + o(n) \;=\; \Gamma\,(A_n,\, \Omega'_k) + o(n).$$

All in all,

$$\Delta_k(n) \;\equiv\; |\Gamma\,(A_n,\, \Omega'_k) - \Gamma\,(B'_n,\, \Omega'_k)| \;\le\; o(n) + c\,\frac{\varepsilon}{\zeta^2(\varepsilon)}\,n.$$

8. The latter inequality implies that, for any $\delta > 0$,

$$\Delta_k(n) \le \delta\, n$$

for all n sufficiently large. This is equivalent to the claim that $\Delta_k(n) = o(n)$.

Thus, Theorem 1.1 is completely proved.

4 Non-Hermitian Toeplitz case

Here we show how Theorem 1.2 follows from Theorem 1.1. To this end, together with the Toeplitz matrices $A_n = A_n(f)$, consider also the optimal circulants $C_n = C_n(f)$.

1. If $f \in L_2$ then $||A_n - C_n||_F^2 = o(n)$ [12, 13].

2. If $f \in L_\infty$ then $||A_n||_2$ and $||C_n||_2$ are upper bounded by $M = \operatorname{ess\,sup} |f|$ (the lowest M for which $|f(t)| \le M$ almost everywhere in the Lebesgue sense).

For A_n, this is seen from the relation [7, 8]

$$(A_n x, y) = \frac{1}{2\pi} \int\limits_{-\pi}^{\pi} f(t) \left(\sum_{k=0}^{n-1} x_k \, e^{ikt} \right) \overline{\left(\sum_{l=0}^{n-1} y_l \, e^{ilt} \right)} \, dt,$$

where $x = [x_0, \ \dots \ , x_{n-1}]^{Я}$, $y = [y_0, \ \dots \ , y_{n-1}]^{Я}$.

For C_n, we know that the eigenvalues of C_n coincide with $(C_n p^{(kn)}, p^{(kn)})$, where $p^{(kn)}$, $1 \le k \le n$, are the columns of the Fourier matrix, and also take into account that [11]

$$(C_n p^{(kn)}, p^{(kn)}) = (A_n p^{(kn)}, p^{(kn)}), \quad 1 \le k \le n.$$

3. C_n are normal matrices, as any circulants are. Moreover, the said above $p^{(kn)}$, $1 \le k \le n$, are the orthonormal eigenvectors for any circulant.

4. Now we need to show that $\lambda(C_n) \sim \Omega$ so long as Ω contains all the values of $f(t)$ (we may change the values on a set with the Lebesgue measure equal to zero).

This is nearly clear when f is a 2π-periodic continuous function. If so, $f(t)$ is uniformly approximated by the Cesaro sums $\sigma_n(t; f)$, and we know that the eigenvalues of $C_n(f)$ coincide with the values of $\sigma_n(t_{kn}; f)$ for $t_{kn} = \frac{2\pi}{n} k$, $1 \le k \le n$. [4, 13]. However, Ω is the eigenvalue cluster for $C_n(f)$ even when $f \in L_1$.

Besides clusters, bring in a more general notion of distribution. Let C_0 stand for the set of all functions which are uniformly continuous and uniformly bounded on the complex plane. We say that complex numbers $\{\lambda_{in}\}_{i=1}^{n}$ are distributed as $f(t)$ if, for any $F \in C_0$,

$$\frac{1}{n} \sum_{i=1}^{n} F(\lambda_{in}) \ \to \ \frac{1}{2\pi} \int\limits_{-\pi}^{\pi} F(f(t)) \, dt.$$

This definition was proposed in [12]; it can be told from the Weyl definition (see [7]) by a larger set of the test functions.

Theorem 4.1 *For any $f \in L_1$, the eigenvalues of the optimal circulants $C_n(f)$ are distributed as $f(t)$.*

To prove this, we can adopt the arguments used for Theorem 4.2 from [13] (with not nearly great changes). By and large, we could do with the observation that, for any $f, g \in L_1$,

$$\sum_{k=1}^{n} |\sigma_n(t_{kn},\, f) - \sigma_n(t_{kn},\, g)| \;\leq\; c\, n\, ||f-g||_{L_1}.$$

The above articles 1–4 match literally the hypotheses (1)–(4) of Theorem 1.1. Clearly, they are satisfied simultaneously as soon as $f \in L_\infty$. To complete the proof of Theorem 1.2, it remains to note that the number of the eigenvalues of $C_n(f)$ falling in any (sufficiently small) ε-neighbourhood of Ω_k is equal to $c_k\, n + o(n)$, where

$$c_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} \chi_k(f(t))dt.$$

This follows from Theorem 4.1 when applied with $F(z) \in C_0$ sufficiently close to $\chi_k(z)$. Indeed, let χ'_k be the characteristic function of Ω'_k, an ε-extension of Ω_k with ε sufficiently small. Then we can choose a nonnegative $F \in C_0$ so that it coincides with χ'_k on Ω'_k and $\operatorname{supp} F$ is larger than $\overline{\Omega'_k}$ yet has no common points with $\overline{\Omega_l}$ for $l \neq k$. On the base of this choice,

$$\begin{aligned} \frac{1}{n}\sum_{i=1}^{n} \chi'_k(\lambda_{in}) \;\leq\; \frac{1}{n}\sum_{i=1}^{n} F(\lambda_{in}) \;&\to\; \frac{1}{2\pi}\int_{-\pi}^{\pi} F(f(t))\, dt \\ &=\; \frac{1}{2\pi}\int_{-\pi}^{\pi} \chi_k(f(t))\, dt. \end{aligned}$$

On the other hand, we can choose a nonnegative $F \in C_0$ so that it coincides with χ_k on Ω_k and $\operatorname{supp} F \subset \Omega'_k$. Then,

$$\begin{aligned} \frac{1}{n}\sum_{i=1}^{n} \chi'_k(\lambda_{in}) \;\geq\; \frac{1}{n}\sum_{i=1}^{n} F(\lambda_{in}) \;&\to\; \frac{1}{2\pi}\int_{-\pi}^{\pi} F(f(t))\, dt \\ &=\; \frac{1}{2\pi}\int_{-\pi}^{\pi} \chi_k(f(t))\, dt. \end{aligned}$$

5 Further results

Consider two disjoint open bounded simply connected fair domains Ω_1 and Ω_2 on the complex plane, and suppose that z is the single common point of their closures. Assume also that there are open bounded simply connected fair domains Ω'_1 and Ω'_2 such that $\overline{\Omega_1} \subset \Omega'_1 \cup \{z\}$, $\overline{\Omega_2} \subset \Omega'_2 \cup \{z\}$, and $\{z\} = \overline{\Omega'_1} \cap \overline{\Omega'_2}$. In addition, we assume that their boundaries are Jordanian curves. We say that Ω_1 and Ω_2 are *separated* (by z) or that z separates Ω_1 and Ω_2, and refer to Ω'_1 and Ω'_2 as (not uniquely determined) *embracing domains*. (Note that $\overline{\Omega_1} \cup \overline{\Omega_2}$ and $\overline{\Omega'_1} \cup \overline{\Omega'_2}$ are also fair domains.)

For a set $\mathcal{M}$ and a point z, if there exist Ω_1 and Ω_2 separated by z and such that $\mathcal{M} \subset \Omega_1 \cup \Omega_2 \cup \{z\}$, then z is said *to separate* $\mathcal{M}$. We refer to Ω_1, Ω_2 and Ω'_1, Ω'_2 as the first and second embracing domains for z and $\mathcal{M}$.

As previously, let $\Gamma(A_n,\, \mathcal{M})$ count how many eigenvalues of A_n belong to $\mathcal{M}$. For any $\delta > 0$, set

$$\mu(\delta, z) = \limsup_{n\to\infty} \frac{\Gamma(A_n,\, D_\delta(z))}{n},$$

where $D_\delta(z)$ is a radius δ disk at z. If $\mu(\delta, z) \to 0$ as $\delta \to 0$, then we call z an *ungreedy point* for $\lambda(A_n)$. All other points are *greedy*, that is, they accumulate too many eigenvalues of A_n.

We are ready now to discern a more detailed structure in clusters.

Theorem 5.1 *Given two sequences of matrices A_n and B_n satisfying the hypotheses (1)–(3) of Theorem 1.1, assume that*

(a) $\lambda(B_n) \sim \Omega_1 \cup \Omega_2$, where Ω_1 and Ω_2 are open bounded simply connected fair domains separated by z_0, and

(b) z_0 is an ungreedy point for $\lambda(B_n)$.

Denote by Ω'_1 and Ω'_2 any embracing domains for Ω_1 and Ω_2 defined through the separation property. Then

$$\Gamma(A_n,\, \Omega'_k) = \Gamma(B_n,\, \Omega'_k) + o(n), \quad k = 1, 2.$$

Proof. Choose $0 < \psi_0 < \pi/2$ and consider the following two sectors:

$$\begin{aligned} S_1^0 &= \{w: \ |w| < 1, \ -\psi_0 < \arg w < \psi_0\}, \\ S_2^0 &= \{w: \ |w| < 1, \ \pi - \psi_0 < \arg w < \pi + \psi_0\}. \end{aligned}$$

Owing to the assumptions made, we can construct a continuous mapping

$$\phi_0 : \ \overline{\Omega_1'} \cup \overline{\Omega_2'} \to \overline{S_1^0} \cup \overline{S_2^0}$$

such that

(a) ϕ_0 is conformal on Ω_1' and Ω_2',

(b) $\phi_0(\overline{\Omega_1'}) = \overline{S_1^0}$ and $\phi_0(\overline{\Omega_2'}) = \overline{S_2^0}$, and

(c) $\phi_0(z) = 0$.

Next, for a given $\varepsilon > 0$, choose a positive α satisfying $\psi \equiv \psi_0 \alpha \le \varepsilon$, and set

$$\phi(z) = \begin{cases} \phi_0^\alpha(z), & z \in \Omega_1'; \\ e^{\mathrm{i}\pi} \left(e^{-\mathrm{i}\pi} \, \phi_0(z)\right)^\alpha, & z \in \Omega_2'. \end{cases}$$

Let us agree that if $z = |z| \, e^{\mathrm{i}\tau}$ then $z^\alpha = |z|^\alpha \, e^{\mathrm{i}\alpha\tau}$.

It is easy to see that ϕ maps $\overline{\Omega_1'} \cup \overline{\Omega_2'}$ continuously onto $\overline{S_1} \cup \overline{S_2}$, where

$$\begin{aligned} S_1 &= \{w: \ |w| < 1, \ -\psi < \arg w < \psi\}, \\ S_2 &= \{w: \ |w| < 1, \ \pi - \psi < \arg w < \pi + \psi\}. \end{aligned}$$

Also,

$$\phi : \ \overline{\Omega_1'} \cup \overline{\Omega_2'} \to \overline{S_1} \cup \overline{S_2}$$

enjoys the following properties (similarly to those of ϕ_0):

(a) ϕ is conformal on Ω_1' and Ω_2',

(b) $\phi(\overline{\Omega_1'}) = \overline{S_1}$ and $\phi(\overline{\Omega_2'}) = \overline{S_2}$, and

(c) $\phi_0(z) = 0$.

Note that ϕ depends on ε. In what follows, we consider ϕ for different sifficiently small ε. It is important for us that in every case

$$|\mathrm{Im}\,\phi(z)| < \varepsilon \quad \forall\, z \in \overline{\Omega_1'} \cup \overline{\Omega_2'}.$$

Take a sufficiently small $\delta > 0$ (independent of $\varepsilon > 0$) and consider the regions $\Omega_k'(\delta) \equiv \Omega_k' \backslash \overline{D_\delta(z)}$ on the z-plane depicted on Fig. 1 and their images $S_k(\delta) \equiv \phi(\Omega_k'(\delta))$ on the w-plane shown on Fig. 2. By the construction of ϕ, $\overline{S_k(\delta)} \subset \overline{\phi_0(\Omega_k'(\delta))}$.

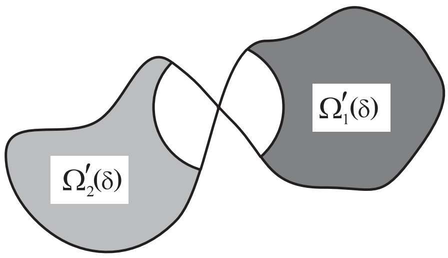

Fig. 1. z-plane.

Using the Keldysh–Mergelyan generalization of the Runge theorem (see [6]), we can approximate ϕ by a polynomial p uniformly on $\overline{\Omega_1'} \cup \overline{\Omega_2'}$. Let $S_k'(\delta) = p(\Omega_k'(\delta))$. If p is sufficiently close to ϕ, then all the points in $\overline{S_k'(\delta)}$ (these are shadowed regions on Fig. 2) are inside or close enough to $\overline{\phi_0(\Omega_k'(\delta))}$.

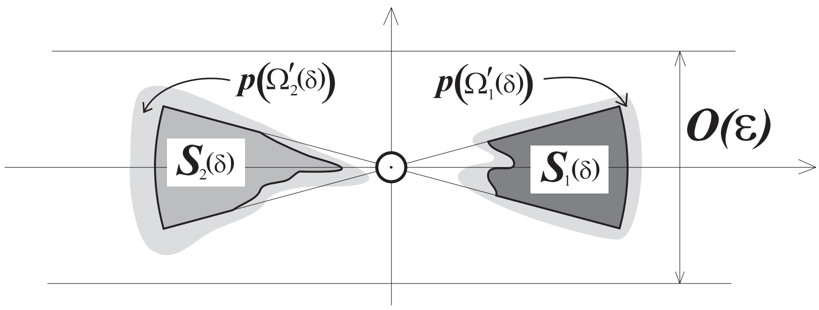

Fig. 2. w-plane.

Consequently, all the points of $S_k'(\delta)$ are separated from $w = 0$ so that there is a disk centered at zero with no common points with $\overline{S_k'(\delta)}$, the shadowed regions on Fig. 2. This disk may depend on δ but it is the same for all sufficiently small ε.

It is clear that there is an open domain $\Omega'' \supset \overline{\Omega'_1} \cup \overline{\Omega'_2}$ such that, first, $p(\Omega'')$ lies inside the ε-tube along the real axis and, second, only $o(n)$ of the eigenvalues of B_n might not belong to Ω''. Since B_n are bounded in the spectral norm uniformly in n, we can transfer to some B'_n for which $||B_n - B'_n||_F^2 = o(n)$ and $\lambda(B'_n) \subset \Omega''$. With no loss of generality, suppose that $B'_n = B_n$. Then

$$||p(A_n) - p(B_n)||_F^2 \; = \; o(n) \qquad \text{and} \qquad \left\| \frac{p(B_n) - (p(B_n))^*}{2} \right\|_2 \; \leq \; \varepsilon.$$

Thus, by Lemma 2.3, the eigenvalues can be indexed so that

$$\sum_{i=1}^{n} |\lambda_i(p(A_n)) - \lambda_i(p(B_n))|^2 \leq c\,\varepsilon\, n.$$

Let $\zeta(\varepsilon)$ be the same as in Article 7 of Section 3, and denote by $S'_k(\delta; \zeta(\varepsilon))$ the $\zeta(\varepsilon)$-extension of $S'_k(\delta)$. It is important that

$$\overline{S'_1(\delta;\zeta(\varepsilon))} \, \cap \, \overline{p(\Omega'_2)} \; = \; \emptyset, \qquad \overline{S'_2(\delta;\zeta(\varepsilon))} \, \cap \, \overline{p(\Omega'_1)} \; = \; \emptyset,$$

for all sufficiently small $\zeta(\varepsilon)$. On the base of the above constructions,

$$\begin{aligned} \Gamma(p(B_n),\, S'_k(\delta)) \;&\leq\; \Gamma(p(A_n),\, S'_k(\delta;\, \zeta(\varepsilon))) + c\,\frac{\varepsilon}{\zeta^2(\varepsilon)}\, n \\ &\leq\; \Gamma(A_n,\, \Omega'_k) + o(n) + c\,\frac{\varepsilon}{\zeta^2(\varepsilon)}\, n. \end{aligned}$$

Furthermore, since

$$\Gamma(B_n,\, \Omega'_k) \leq \Gamma(p(B_n),\, S'_k(\delta)) + \mu(\delta, z)\, n + o(n)$$

and z is not an accumulation point for B_n, we obtain eventually that

$$\Gamma(B_n, \Omega'_k) \;\leq\; \Gamma(A_n, \Omega'_k) + o(n), \quad k = 1, 2.$$

Using these inequalities along with (since $\Omega_1 \cup \Omega_2$ is a fair domain, we can apply Theorem 1.1)

$$\Gamma(B_n, \Omega'_1) + \Gamma(B_n, \Omega'_2) \;=\; \Gamma(A_n, \Omega'_1) + \Gamma(A_n, \Omega'_2) + o(n),$$

we conclude that

$$\Gamma(B_n,\, \Omega_k') \;=\; \Gamma(A_n,\, \Omega_k') + o(n), \quad k = 1, 2,$$

which completes the proof. $\square$

For a point z and $\delta > 0$, denote by $\mu(\delta, z;\, f)$ the Lebesgue measure of the preimage of the disk $D_\delta(z)$. We call $z = f(t)$ an *essential point* for f if $\mu(\delta, z;\, f) > 0$ for any $\delta > 0$. If $\mu(\delta, z;\, f) \to 0$ as $\delta \to 0$, then z is said to be an *ungreedy point* for f.

Theorem 5.2 *Denote by $\mathcal{M}$ the set of the essential values for $f \in L_\infty$. Assume that z is an ungreedy point for f that separates $\mathcal{M}$ with the embracing domains Ω_1 and Ω_2, and consider the Toeplitz matrices $A_n = A_n(f)$. Then*

$$\lim_{n\to\infty} \frac{\Gamma(A_n,\, \Omega_k)}{n} \;=\; \frac{1}{2\pi} \int\limits_{-\pi}^{\pi} \chi_k(f(t))\, dt, \quad k = 1, 2,$$

where χ_k is the characteristic function for Ω_k.

This theorem stems naturally from Theorem 5.1 with the optimal circulants $C_n(f)$ serving as B_n. The details can be gleaned from Section 4.

In the rest of this section, we propose some further research steps to be done. As above, let $\mathcal{M}$ stand for the set of all essential points for a symbol f. Let us say that $f(t)$ is an *open curve in* Ω if every point of $\mathcal{M} \cap \Omega$ is a separation point for $\mathcal{M}$. According to this definition, we admit that the points $f(t)$ for $f(t) \notin \Omega$ might not separate $\mathcal{M}$. Thus, the whole of $f(t)$ for all t is not necessarily an open curve. We propose the following theorem.

Let $f \in L_\infty$ generate an open curve in an open bounded simply connected fair domain Ω. Then, for the Toeplitz matrices $A_n = A_n(f)$, the Szego-like formula

$$\frac{1}{n} \sum_{i=1}^{n} F(\lambda_i(A_n)) \;\to\; \frac{1}{2\pi} \int\limits_{-\pi}^{\pi} F(f(t))\, dt$$

is valid for any $F \in C_0$ supported inside Ω.

We intend to prove this in another paper.

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