Multilevel Toeplitz matrices and approximation by matrix algebras

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ABSTRACT

Optimal preconditioners (those that provide a proper cluster at 1) are of paramount importance for cg-like methods since they make them converge superlinearly. In preceding papers, we proved that any preconditioner belonging to *partially equimodular* spaces[22] is not optimal for multilevel Toeplitz matrices where the aforementioned class of spaces includes all the known and used trigonometric matrix algebras. Here we survey and refine these results by focusing our attention on the (surprisingly) more difficult case in which the multilevel Toeplitz matrices are Hermitian.

Key words: Multilevel Toeplitz matrices, preconditioning and clustering, conjugate gradient, matrix algebras.

1 A LONG PRELUDE

This work was bred in a simple observation that *in the multilevel case optimal preconditioners may be not optimal.* This striking claim contains, of course, two different concepts of optimality. The first is in T.Chan's sense: a preconditioner C_n is called optimal for A_n if it minimizes $||A_n - C_n||_F$ over some appropriate set of matrices C_n. Usually this set is a matrix algebra; in[5] it was the set of circulant matrices. Such a preconditioner is easy to construct, as the Frobenius norm is expressed through the matrix entries in the most friendly way. Moreover, we show below that this preconditioner can never be *very bad* among the given possibilities to select from. The second concept of optimality indicates that the preconditioned eigenvalues (singular values) are clustered at one, and the cluster is *proper* in the sense of.[25,24] To recall the definitions, consider matrices C_n and A_n, both of order

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