Pseudo-Skeleton Approximations

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Abstract

Let an $m \times n$ matrix A be approximated by a rank r matrix with an accuracy ε. The paper addresses the problem of construction of accuracy estimates of the so called pseudo-skeleton approximations using r columns and r rows of the matrix to be approximated. We derive the upper bound accuracy estimate of the form $\mathcal{O}(\varepsilon \sqrt{r}\,(\sqrt{m}+\sqrt{n}))$ in the sense of the 2-norm.

1 Introduction

Let $A \in \mathbb{R}^{m\times n}$ and assume that rank $A = r$. Then there exists a nonsingular $r \times r$ submatrix $\hat{A}$ in A. Denote the columns and rows of A containing the submatrix $\hat{A}$ by $C \in \mathbb{R}^{m\times r}$ and $R \in \mathbb{R}^{r\times n}$, respectively. It is easy to verify that

$$A = C\hat{A}^{-1}R. \tag{1}$$

This decomposition is known as a skeleton decomposition of A.

Now let us suppose that rank $A \approx r$ means that rank $(A + E) = r$, where $E \approx 0$ in the sense of a prescribed matrix norm. The exact equality rank $A = r$ implies the exact equality (1) and we wonder if the approximate equality rank $A \approx r$ may imply the approximate equality

$$A \approx B = CGR, \tag{2}$$

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where $G \in \mathbb{R}^{r\times r}$ is not necessarily equal to $\hat{A}^{-1}$ and even not necessarily nonsingular. The matrix B in (2) will be called the pseudo-skeleton component of the matrix A.

Assume that A possesses an accurate enough (say, accurate to within $\varepsilon > 0$) rank r matrix approximation. The question arises how accurately such an A can be approximated by one of its pseudo-skeleton components? The answer is given by

Theorem 1. *Assume that $A,\, F \in \mathbb{R}^{m\times n}$, $\operatorname{rank}(A-F) \le r$, and $\|F\|_2 \le \varepsilon$, for some $\varepsilon > 0$. Then there exist r columns and r rows in A which determine a pseudo-skeleton component CGR such that*

$$\|A - CGR\|_2 \le \varepsilon \left(1 + (\sqrt{t(r,n)} + \sqrt{t(r,m)})^2 \right), \tag{3}$$

where

$$t(r,n) = \frac{1}{\min\limits_{U} \max\limits_{P \in \mathcal{M}(U)} \sigma_{min}(P)}; \tag{4}$$

$$U^T U = I,\; U \in \mathbb{R}^{n\times r},\; r \le n; \tag{5}$$

by $\mathcal{M}(U)$ we denote the set of all $r \times r$ submatrices in U; $\sigma_{min}(P)$ is the minimal singular value of P.

Corollary. *Under the hypotheses of Theorem 1 there exists a pseudo-skeleton component such that*

$$\|A - CGR\|_2 \le \varepsilon \left(1 + 2\sqrt{rn} + 2\sqrt{rm}\, \right). \tag{6}$$

This bound immediately follows from the following nontrivial inequality:

$$t(r,n) \le \sqrt{r(n-r) \;+\; \min\{r, n-r\}}. \tag{7}$$

This inequality was recently obtained in [1], [2].

It becomes the equality in the two extreme cases: $r = 1$ and $r = n-1$. In other cases this estimate is not sharp. We have a conjecture that the inequality

$$t(r,n) \le \sqrt{n}$$

holds true. At least, we do not know any matrix for which it is violated.

We would like to emphasize that Theorem 1 is somewhat different from theorems of the small perturbations theory. If $\varepsilon \to 0$ then estimate (3) can be significantly improved. However, in the most interesting and important cases ε may depend on m and n, and usually the decrease of ε corresponds to the increase of the matrix size.

We have a proof that is almost constructive and involves two stages:

(a) the choice of appropriate C and R;

(b) the choice of G.

Both stages of the proof make use of the explicit knowledge of F. This is prohibitive from the practical point of view, because usually we know nothing about F except that it exists. Exploiting another choice of G with no explicit information of $\hat{F}$ leads to a more coarse estimate.

Theorem 2. *Under assumptions of Theorem 1 there exists G which can be chosen using only $\hat{A}$ and which provides the estimate*

$$\|A - CGR\|_2 \leq \varepsilon\sqrt{(1 + t^2(r,p)\ )}\left(1 + (\sqrt{t(r,n)} + \sqrt{t(r,m)})^2\ \right), \tag{8}$$

where $p = \min(m,n)$.

Proof of the Theorem 1. Consider the decomposition

$$A - F = U \Sigma V,$$

where $\Sigma = \operatorname{diag}(\sigma_1, \ldots, \sigma_r)$, $\sigma_1 \geq \ldots \geq \sigma_r \geq 0$; $U^T U = V V^T = I$, and submatrices $\hat{U}$, $\hat{V} \in \mathbb{R}^{r \times r}$ of U, V, respectively, such that

$$\|\hat{U}^{-1}\|_2 \leq t(r,m), \qquad \|\hat{V}^{-1}\|_2 \leq t(r,n). \tag{9}$$

We now select the r rows and r columns determined by the choice of $\hat{U}$ and $\hat{V}$, respectively.

Let C and F_C denote $m \times r$ submatrices, R and F_R denote $r \times n$ submatrices of A and of F, respectively, which correspond to the selected rows and columns. Let $\hat{A}$ and $\hat{F}$ denote the $r \times r$ submatrices which occupy the intersection of these rows and columns in A and F. Then we have

$$\begin{aligned} CGR &= (U\Sigma\hat{V} + F_C)G(\hat{U}\Sigma V + F_R) \\ &= U\hat{U}^{-1}(\Phi G \Phi)\hat{V}^{-1}V + E, \end{aligned} \tag{10}$$

where

$$E = U\hat{U}^{-1}(\Phi G)F_R + F_C(G\Phi)\hat{V}^{-1}V + F_C G F_R, \tag{11}$$

$$\Phi = \hat{U}\Sigma\hat{V} = \hat{A} - \hat{F}. \tag{12}$$

Now consider the singular value decomposition of Φ:

$$\Phi = \tilde{U}\tilde{\Sigma}\tilde{V}, \quad \tilde{\Sigma} = \mathrm{diag}\,(\tilde{\sigma}_1, \ldots, \tilde{\sigma}_r), \quad \tilde{U}^T\tilde{U} = \tilde{V}^T\tilde{V} = I.$$

Let $\tau > 0$ be a threshold value which will be specified later. Introducing the notation

$$\tilde{\Sigma}_\tau \equiv \mathrm{diag}\,(\tilde{\sigma}_i) = \begin{cases} \sigma_i, & \text{if } \sigma_i \geq \tau, \\ 0, & \text{otherwise;} \end{cases}$$

$$\tilde{\Sigma}_\tau^+ \equiv \mathrm{diag}\,(\tilde{\sigma}_i^+) = \begin{cases} \sigma_i^{-1}, & \text{if } \sigma_i \geq \tau, \\ 0, & \text{otherwise.} \end{cases}$$

$$\Phi_\tau = \tilde{U}\tilde{\Sigma}_\tau\tilde{V}, \qquad \Phi_\tau^+ = \tilde{V}^T\tilde{\Sigma}_\tau^+\tilde{U}^T,$$

we see that

$$\Phi\Phi_\tau^+\Phi = \Phi_\tau, \qquad \|\Phi\Phi_\tau^+\|_2 \leq 1, \qquad \|\Phi_\tau^+\Phi\|_2 \leq 1. \tag{13}$$

If we set

$$G = \Phi_\tau^+, \tag{14}$$

then relations (11) and (13) imply that

$$\|E\|_2 \leq \varepsilon\left(\|\hat{U}^{-1}\|_2 + \|\hat{V}^{-1}\|_2 + \frac{\varepsilon}{\tau}\right). \tag{15}$$

Note that $A - F = U\hat{U}^{-1}\Phi\hat{V}^{-1}V$. Using this equality in conjunction with (10)–(15) we get the estimate

$$\|A - CGR\|_2 \leq \varepsilon + \tau\|\hat{U}^{-1}\|_2\|\hat{V}^{-1}\|_2 + \frac{\varepsilon^2}{\tau} + \varepsilon\|\hat{U}^{-1}\|_2 + \varepsilon\|\hat{V}^{-1}\|_2. \tag{16}$$

Now setting

$$\tau = \varepsilon/\sqrt{\|\hat{U}^{-1}\|_2\,\|\hat{V}^{-1}\|_2}$$

and substituting for (9) we complete the proof of the theorem.

Proof of the Theorem 2. Without loss of generality we assume that the matrix A is of the form

$$A = U\Sigma V \equiv U \begin{bmatrix} \Sigma_1 & 0 \\ 0 & \Sigma_2 \end{bmatrix} V \in \mathbb{R}^{m\times n},$$

where $m \leq n$, $\Sigma_1 \in \mathbb{R}^{r\times r}$, $\Sigma_2 \in \mathbb{R}^{(m-r)\times(n-r)}$, $\|\Sigma_2\|_2 \leq \varepsilon$, and the leading $r \times r$ submatrices in U and V satisfy the inequalities (9).

Take the partitioning of U which is induced by that of Σ

$$U = \begin{bmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{bmatrix}, \qquad U_{11} \in \mathbb{R}^{r\times r}, \qquad U_{22} \in \mathbb{R}^{(m-r)\times(m-r)},$$

and consider the matrix

$$\tilde{F} = U \begin{bmatrix} 0 & E \\ 0 & \Sigma_2 \end{bmatrix} V, \qquad E = -U_{11}^{-1}\,U_{12}\,\Sigma_2.$$

Obviously, $rank(A - \tilde{F}) \leq r$ and

$$\|\tilde{F}\|_2^2 = \|E^T E + \Sigma_2^T\Sigma_2\|_2 \leq \|E\|_2^2 + \|\Sigma_2\|_2^2 \leq \varepsilon^2\ \left(1 + t^2(r,n)\right).$$

Now apply Theorem 1 to the matrices A and $\tilde{F}$. The first r rows of the matrix $\tilde{F}$ are by construction zero; therefore G will be computed using the submatrix $\tilde{A} - \tilde{F} = \tilde{A}$, ie. using only the entries of A. The inequality (8) is thus proven for $p = m$; applying the same train of reason to the matrix A^T, we arrive at (8) for $p = n$.

References

[1] S. Chandrasekaran and I. Ipsen, *On Rank-Revealing QR Factorizations.* Research Report YALEU/DCS/RR-880, December 1991.

[2] Y. P. Hong and C.-T. Pan. *Rank-revealing QR factorizations and singular value decomposition.* Math. Comput., 58(1992), no. 197, pp. 213–232.