A general equidistribution theorem for the roots of orthogonal polynomials *

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ABSTRACT

It is well-known that the roots of any two orthogonal polynomials are distributed equally if the weights satisfy the Szegö condition. In this paper, we propose a general equidistribution theorem that does not use the Szegő condition and admits an elementary proof.

Key-words: orthogonal polynomials, three-term recurrences, polynomial roots, equidistribution, Szegő condition.

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1 Introduction

In many questions of mathematics, given a sequence of n-point sets $\{x_{in}\}_{i=1}^{n}$, we are interested to study the asymptotic behavior of these sets as $n \to \infty$. A prolific concept for this is one of equidistribution.

Let x_{in} and $\tilde{x}_{in}$ be real (or complex) numbers. Then we write

$$x_{in} \sim \tilde{x}_{in}$$

and call $\{x_{in}\}_{i=1}^{n}$ and $\{\tilde{x}_{in}\}_{i=1}^{n}$ *equidistributed* if

$$\lim_{n\to\infty} \frac{1}{n} \left(\sum_{i=1}^{n} F(x_{in}) - \sum_{i=1}^{n} F(\tilde{x}_{in}) \right) = 0 \tag{1}$$

for any function $F(x)$ from a prescribed set $\mathcal{F}$. A convenient and general enough choice is when $\mathcal{F}$ consists of all continuous functions with a bounded support [11].

This definition played the key role in recent simplifications in the proof and various generalizations of the Szegő formula [3] for the eigenvalues of Hermitian Toeplitz matrices [11, 13]. It was successfully used in a lot of papers on the spectra of structured matrices (see [7, 10]); a nice sketch and guide in the related fields is [1].

The same definition allows one to describe the asymptotic behavior of the roots of polynomials orthogonal on an interval.

Consider two sequences of polynomials, $p_n(x)$ and $\tilde{p}_n(x)$, with real coefficients and the highest degree coefficient positive, and assume that they are orthogonal in the following sense:

$$(p_m, p_n)_\sigma = \delta_{mn}, \quad (\tilde{p}_m, \tilde{p}_n)_{\tilde{\sigma}} = \delta_{mn},$$

where

$$(f, g)_\sigma = \int_{-1}^{1} f(x)g(x)d\sigma(x), \quad (f, g)_{\tilde{\sigma}} = \int_{-1}^{1} f(x)g(x)d\tilde{\sigma}(x),$$

$\sigma(x)$ and $\tilde{\sigma}(x)$ are monotone functions with infinitely many points of growth; δ_{mn} is 1 for $m = n$ and 0 otherwise. Note that we have uniquely determined infinite sequences of orthogonal polynomials [2, 9].

Denote by x_{in} and $\tilde{x}_{in}$ the roots of $p_n(x)$ and $\tilde{p}_n(x)$, respectively. Then $x_{in} \sim \tilde{x}_{in}$ provided that both σ and $\tilde{\sigma}$ satisfy the following *Szegő condition*:

$$\int_{-1}^{1} \frac{\log \sigma'(x)}{\sqrt{1-x^2}} dx > -\infty, \quad \int_{-1}^{1} \frac{\log \tilde{\sigma}'(x)}{\sqrt{1-x^2}} dx > -\infty. \tag{2}$$

The proofs given in [2, 9] are far from trivial. Moreover, within the approach therein, the fact itself is far from evident even under more restrictive assumptions on σ and $\tilde{\sigma}$.

One straightforward proof has been recently suggested in [12]. It begins with the elementary observation that the roots coincide with the eigenvalues of the Hermitian tridiagonal matrices made up of the coefficients of the three-term recurrences for the orthogonal polynomials:

$$\begin{aligned} x p_n(x) &= b_{n-1} p_{n-1}(x) + a_n p_n(x) + b_n p_{n+1}(x), \\ x \tilde{p}_n(x) &= \tilde{b}_{n-1} \tilde{p}_{n-1}(x) + \tilde{a}_n \tilde{p}_n(x) + \tilde{b}_n \tilde{p}_{n+1}(x). \end{aligned}$$

In matrix notation, these read

$$\begin{aligned} x[p_0(x), \ldots, p_n(x)] &= [p_0(x), \ldots, p_n(x)] A_n + [0 \ldots 0 \; b_n] p_{n+1}(x), \\ x[\tilde{p}_0(x), \ldots, \tilde{p}_n(x)] &= [\tilde{p}_0(x), \ldots, \tilde{p}_n(x)] \tilde{A}_n + [0 \ldots 0 \; \tilde{b}_n] \tilde{p}_{n+1}(x), \end{aligned}$$

where

$$A_n = \begin{bmatrix} a_0 & b_0 & & & \\ b_0 & a_1 & b_1 & & \\ & \ddots & \ddots & \ddots & \\ & & b_{n-2} & a_{n-1} & b_{n-1} \\ & & & b_{n-1} & a_n \end{bmatrix}, \tag{3}$$

$$\tilde{A}_n = \begin{bmatrix} \tilde{a}_0 & \tilde{b}_0 & & & \\ \tilde{b}_0 & \tilde{a}_1 & \tilde{b}_1 & & \\ & \ddots & \ddots & \ddots & \\ & & \tilde{b}_{n-2} & \tilde{a}_{n-1} & \tilde{b}_{n-1} \\ & & & \tilde{b}_{n-1} & \tilde{a}_n \end{bmatrix}. \tag{4}$$

It is obvious now that the roots of $p_{n+1}(x)$ are the eigenvalues of A_n while those of $\tilde{p}_{n+1}(x)$ are the eigenvalues of $\tilde{A}_n$.

Then, we make use of the well-known limiting relations for a_n and b_n:

$$\lim_{n\to\infty} a_n = 0, \quad \lim_{n\to\infty} b_n = \frac{1}{2}. \tag{5}$$

It follows immediately that A_n and $\tilde{A}_n$ are asymptotically close in the following sense:

for any $\varepsilon > 0$ *there exists a splitting* $A_n - \tilde{A}_n = E_{\varepsilon n} + R_{\varepsilon n}$
with $||E_{\varepsilon n}||_F^2 \le \varepsilon n, \quad \mathrm{rank} R_{\varepsilon n} \le \varepsilon n.$

As we know from [11], such splittings lead to equidistribution of the eigenvalues of A_n and $\tilde{A}_n$ (if the matrices are non-Hermitian, the equidistribution holds for the singular values).

The only place that is not quite elementary in the above proof is (5). And this is the only place where we use the Szegő condition. A proof can be found in [2, 9]; a direct way to (5), still chiefly along the same ideas, is sketched in [12].

From the equidistribution property we readily infer that any σ, under the Szegő condition, generates the orthogonal polynomials with the roots distributed equally with the roots of the Chebyshev polynomials.

If the Szegő condition does not hold, then the limiting relations (5) can be lost. In some particular cases (see [14]) we can have them in a modified form (for subsequences). For such cases we are able to adopt our matrix-based proof. However, the limiting relations in the above or modified form remain a rather nontrivial part of the enterprise.

Thus, on the way to an approach elementary in every detail, one might think about proving the equidistribution property without any reference to the limiting relations like (5). This is the main result of this paper.

At the same time, we present a *new* equidistribution theorem that covers some cases where the Szegő condition is not fulfilled. We discover that equidistribution of the roots takes place whenever σ and $\tilde{\sigma}$ enjoy the following *equivalence property*:

$$m\ (f,f)_\sigma \ \le\ (f,f)_{\tilde{\sigma}} \ \le M\ \ (f,f)_\sigma, \tag{6}$$

where m and M are positive constants, and f is an arbitrary polynomial. (In spite of the generality, the equivalence property is certainly a restriction that is still to be better understood.)

Some recent references on the subject with a classical approach using complex analysis tools are [**?**, 4], for a matrix approach see also [5].

2 Auxiliary lemma

Lemma 2.1 *Given two sequences of $n \times n$ matrices, A_n and H_n, assume that, for all n,*

$$H_n^* = H_n$$

and

$$\lim_{n\to\infty} \frac{||A_n - H_n||_F^2}{n} \;=\; 0.$$

Then the eigenvalues of A_n and H_n are equidistributed.

This lemma is a consequence of Lemma 2.3 from [13]. For the reader's convenience, below we give a direct proof.

Proof. We use the following fact: if A_n is an arbitrary matrix and H_n is Hermitian, then their eigenvalues can be ordered such that [8] (for a short proof, see also [13])

$$\sum_{i=1}^{n} |\lambda_i(A_n) - \lambda_i(H_n)|^2 \leq 2\; ||A_n - H_n||_F^2.$$

As we know from [11], this is sufficient for the equidistribution of $\lambda_i(A_n)$ and $\lambda_i(H_n)$. $\square$

The eigenvalues of A_n can be complex numbers. However, in the constructions that follow these will be still real numbers.

3 Main theorem

Theorem 3.1 *Assume that σ and $\tilde{\sigma}$ are equivalent in the sense of (6). Then the roots of the corresponding orthogonal polynomials are equidistributed in the sense of (1).*

Proof. Note that, for any n, the system of polynomials $P_0(x), \ldots, P_n(x)$, where $P_i(x)$ can be $p_i(x)$ or $\tilde{p}_i(x)$ (the choice may alter for different i), is linearly independent, and consider the following expansions:

$$
\begin{aligned}
x\ p_n(x) &= \sum_{i=0}^{n} r_i\ p_i(x) + r_{n+1}\ \tilde{p}_{n+1}(x), \\
x\ \tilde{p}_{n+1}(x) &= \sum_{i=0}^{n} t_i\ p_i(x) + \tilde{a}_{n+1}\ \tilde{p}_{n+1}(x) + \tilde{b}_{n+1}\ \tilde{p}_{n+2}(x), \\
x\ \tilde{p}_{n+2}(x) &= \tilde{b}_{n+1}\ \tilde{p}_{n+1}(x) + \tilde{a}_{n+2}\ \tilde{p}_{n+2}(x) + \tilde{b}_{n+2}\ \tilde{p}_{n+3}(x).
\end{aligned}
$$

In matrix notation, we obtain

$$
\begin{aligned}
& x\ [p_0(x),\ \ldots\ ,\ p_n(x),\ \tilde{p}_{n+1}(x),\ \tilde{p}_{n+2}(x)] \\
= {} & [p_0(x),\ \ldots\ ,\ p_n(x),\ \tilde{p}_{n+1}(x),\ \tilde{p}_{n+2}(x)]\ B_{n+2} \quad + \quad [0 \ldots 0\ \tilde{b}_{n+2}]\ \tilde{p}_{n+3}(x),
\end{aligned}
$$

where

$$
B_{n+2} = \begin{bmatrix}
a_0 & b_0 & & & & & \\
b_0 & a_1 & b_1 & & & & \\
 & \ddots & \ddots & \ddots & & & \\
 & & b_{n-2} & a_{n-1} & b_{n-1} & & \\
r_0 & \ldots & r_{n-2} & r_{n-1} & r_n & r_{n+1} & \\
t_0 & \ldots & t_{n-2} & t_{n-1} & t_n & \tilde{a}_{n+1} & \tilde{b}_{n+1} \\
 & & & & & \tilde{b}_{n+1} & \tilde{a}_{n+2}
\end{bmatrix}.
$$

Therefore, the roots of $\tilde{p}_{n+3}(x)$ coincide with the eigenvalues of B_{n+2} (and with those of $\tilde{A}_{n+2}$ defined by (4)). As we see from the introduction, the roots of $p_{n+3}(x)$ coincide with the eigenvalues of A_{n+2} defined by (3).

The next step is to show that

$$
||B_{n+2} - A_{n+2}||_F^2 = O(1). \tag{7}
$$

By simple calculation, we have

$$
||B_{n+2} - A_{n+2}||_F^2 \ \leq\ 2(c_1 + c_2),
$$

where

$$
\begin{aligned}
c_1 &= a_n^2 + a_{n+1}^2 + a_{n+2}^2 + b_{n-1}^2 + 2b_n^2 + 2b_{n+1}^2. \\
c_2 &= \sum_{i=0}^{n+1} r_i^2 + \sum_{i=0}^{n} t_i^2 + \tilde{a}_{n+1}^2 + \tilde{a}_{n+2}^2 + 2\tilde{b}_{n+1}^2.
\end{aligned}
$$

It is easy to see that all the coefficients of the three-term recurrences are bounded from above by 1. Indeed, since $|x| \leq 1$, because of the orthogonality conditions and due to the Cauchy inequality,

$$\begin{aligned} |a_n| &= |(xp_n, p_n)_\sigma| \leq 1, \\ |b_n| &= |(xp_n, p_{n+1})_\sigma| \leq \sqrt{(xp_n, xp_n)_\sigma \ (p_{n+1}, p_{n+1})_\sigma} \leq 1. \end{aligned}$$

Hence, it is sufficient to establish the boundedness of the two sums with the quantities r_i^2 and t_i^2.

Taking into account the equivalence property (6) and using the Cauchy inequality, we obtain

$$|r_{n+1}| = |(xp_n, \tilde{p}_{n+1})_{\tilde{\sigma}}| \leq \sqrt{(xp_n, xp_n)_{\tilde{\sigma}} \ (\tilde{p}_{n+1}, \tilde{p}_{n+1})_{\tilde{\sigma}}} \leq \sqrt{M}.$$

Further,

$$\begin{aligned} \sum_{i=0}^{n} r_i^2 &= \left(\sum_{i=0}^{n} r_i p_i, \ \sum_{i=0}^{n} r_i p_i \right)_\sigma = (xp_n - r_{n+1}\tilde{p}_{n+1}, \ xp_n - r_{n+1}\tilde{p}_{n+1})_\sigma \\ &\leq 2 \ (xp_n, xp_n)_\sigma + 2 \ r_{n+1}^2 \ (\tilde{p}_{n+1}, \tilde{p}_{n+1})_\sigma \leq 2 \ \left(1 + \frac{M}{m}\right). \end{aligned}$$

Consider now the sum with t_i^2. From the uniqueness of the above expansions,

$$\sum_{i=0}^{n} t_i p_i(x) = \tilde{b}_n \tilde{p}_n(x).$$

Consequently,

$$\sum_{i=0}^{n} t_i^2 = (\sum_{i=0}^{n} t_i p_i, \ \sum_{i=0}^{n} t_i p_i)_\sigma = \tilde{b}_n^2 \, (\tilde{p}_n, \tilde{p}_n)_\sigma \leq \frac{1}{m},$$

and this completes the proof of (7).

To finish the proof of the theorem, apply Lemma 2.1 to the Hermitian matrices A_{n+2} and to the non-Hermitian matrices B_{n+2}. The premises of this lemma are obviously fulfilled as we have (7). □

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