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Spectra of Multilevel Toeplitz Matrices: Advanced Theory via Simple Matrix Relationships

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ABSTRACT

We consider the eigen and singular value distributions for m-level Toeplitz matrices generated by a complex-valued periodic function f of m real variables. We show that familiar formulations for $f \in L_\infty$ (due to Szego and others) can be preserved so long as $f \in L_1$, and what is more, with G. Weyl's definitions to be just a bit changed. In contrast to other approaches, the one we follow in this paper is based on simple matrix relationships.

1 Introduction

Multilevel Toeplitz matrices arise naturally in multidimensional Fourier analysis. Given a complex-valued function f of m real variables which is 2π-periodic in each of them, assume that f is L_1-integrable on a cube $\Pi^m = [0, 2\pi]^m$ and associate it with a Fourier series

$$f(x) \sim \sum_{k \,\in\, \mathbb{Z}^m} a_k e^{\mathrm{i}(k,x)},$$

where

$$k = (k_1, \ldots, k_m), \quad x = (x_1, \ldots, x_m), \quad (k, x) = k_1 x_1 + \ldots + k_m x_m.$$

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The multilevel (to be precise, m-level) Toeplitz matrices are introduced as follows:

$$A_n = [a_{k-l}],$$

$$k = (k_1, \ldots, k_m), \ l = (l_1, \ldots, l_m), \ n = (n_1, \ldots, n_m),$$

$$0 \le k_j, l_j \le n_j - 1, \quad j = 1, \ldots, m.$$

The multiindices k and l satisfying the last inequalities will be referred to as n-admissible. For further reference, recall that A_n becomes a multilevel (m-level) circulant if

$$a_{k-l} = a_{k-l(\operatorname{mod} n)},$$

where, by definition,

$$k(\operatorname{mod} n) \equiv (k_1(\operatorname{mod} n_1), \ \ldots, k_m(\operatorname{mod} n_m)).$$

The matrices A_n and C_n can be viewed as ones with a nested block structure: A_n consists of $n_1 \times n_1$ blocks, every block consists of $n_2 \times n_2$ smaller ones, and so on. Obviously,

$$A_n \in \mathbb{C}^{N \times N}, \quad \text{where} \quad N = N(n) \equiv n_1 \ldots n_m.$$

The classical Szego theorem (see [2]) considers the case $m = 1$, and states that for any real-valued periodic function $f \in L_\infty$ the eigenvalues of the (Hermitian) Toeplitz matrices A_n are asymptotically distributed as the values of $f(x)$. The same is known as well in the case of arbitrarily many dimensions (see [2, 4]).

Keeping in mind these well-known facts, one might be interested to ask whether the same stands under relaxed requirements to f: for example, when $f \in L_2$ or even when $f \in L_1$.

Prior to thinking it over, we need to recollect the definitions. Let i and n denote the multiindices

$$i = (i_1, \ldots, i_m), \quad n = (n_1, \ldots, n_m).$$

Due to G. Weyl, a sequence of suites

$$\{\lambda_{in}\}, \quad 1 \le i_j \le n_j, \quad j = 1, \ldots, m,$$

is called distributed as $f(x)$ if it enjoys the following properties:

(1) there exist numbers $m \le M$ such that $m \le \lambda_{in} \le M$ for all i, n;

(2) for any continuous on $[m, M]$ function $F(x)$,

$$\lim_{n \to \infty} \frac{\sum\limits_{i=1}^{N(n)} F(\lambda_{in})}{N(n)} = \frac{1}{(2\pi)^m} \int\limits_{\Pi^m} F(f(x))dx. \qquad (*)$$

Here and further on, the multiindex tending to infinity is meant in the sense that every its component tends to infinity:

$$n = (n_1, \ldots, n_m) \to \infty \quad \Leftrightarrow \quad n_j \to \infty, \ j = 1, \ldots, m.$$

If $f \in L_2$ then it does not follow from anywhere that the eigenvalues of A_n for all n belong to a common finite interval. Thus, to begin with *we should modify the definition.*

Let us adopt the definition proposed in [6, 8]. A sequence of suites $\{\lambda_{in}\}$ will be termed distributed as $f(x)$ if $(*)$ holds for any continuous function $F(x)$ with a finite support. In this case, we shall write $\lambda_{in} \sim f(x)$. If the eigen or singular values of matrices A_n are distributed as $f(x)$, we reflect this by writing $\lambda(A_n) \sim f(x)$ or $\sigma(A_n) \sim f(x)$, respectively.

It is proved in [6, 8] that if $f(x) \in \mathbb{R}$ and $f \in L_2$ then $\lambda(A_n) \sim f(x)$, while if $f(x) \in \mathbb{C}$ and $f \in L_2$ then $\sigma(A_n) \sim |f(x)|$ (for $f \in L_\infty$ it was proved in [1, 3]). We now put a question: does the same hold for $f \in L_1$?

Note yet that answering this question is only a formal purpose. Above all, we want to present a new technique based on some easily detectable matrix relationships.

Concerning the formal purpose, we should compare what we do with some constructions from [2] that allow one, in principle, to treat those f that are not necessarily in L_∞. To this end, of course, there were some new notions (besides G. Weyl's definition) put forth in [2]. However, they were used only in a new elegant proof of an analogue of the Szego theorem. Natural interrelations between Toeplitz matrices and circulants discovered there seemed to be lost with L_∞ replaced by L_1. In this paper these relations are rejuvenated with the help of low-rank correction matrices.

To prove that $\lambda(A_n) \sim f(x)$ (for real-valued f) we can build up a sequence of multilevel circulants C_n such that

$$\lambda(A_n) \sim \lambda(C_n) \tag{1}$$

and, simultaneously,

$$\lambda(C_n) \sim f(x). \tag{2}$$

Two sequences of suites $\{\lambda_{in}\}$ and $\{\mu_{in}\}$ are referred to as equially distributed if for any continuous function $F(x)$ with a finite support

$$\lim_{n \to \infty} \frac{\sum_{i=1}^{N(n)} (F(\lambda_{in}) - F(\mu_{in}))}{N(n)} = 0.$$

Let C_n be an optimal multilevel circulant (see [5]) for A_n, the one that minimizes $||A_n - C_n||_F$ over all multilevel circulants. Throught the paper, if $n = (n_1, \ldots, n_m)$ then let $\alpha_n = o(n)$ signify that $\alpha_n / N(n) \to 0$ as $n \to \infty$. Then it can be proved [6] that

$$f \in L_2 \quad \Rightarrow \quad \|A_n - C_n\|_F^2 = o(n) \quad \Rightarrow \quad (1).$$

It can be also proved [6] that

$$f \in L_1 \quad \Rightarrow \quad \lambda(C_n) \sim f(x).$$

In the case $f \in L_1$ we are not aware yet how to prove the property (1). In contrast to the L_2 case, the key property (that entailed (1) previously)

$$\|A_n - C_n\|_F^2 = o(n)$$

is no longer valid for an arbitrary $f \in L_1$ [9]. However, it will be shown below that the above equation can be saved after "low-rank" corrections of the matrices involved. It appears to be sufficient for the distribution results under question to follow quite easily.

2 Equally Distributed Sequences

If $f \in L_1$ then we ought to keep in mind that the multilevel Toeplitz matrices A_n can be not "close by norm" to the corresponding optimal multilevel circulants C_n. All the same, we will prove that these matrices are still "close" but in a somewhat different sense. Results cited below suggest how to enrich the standard view of closeness.

Theorem 1 [7]. *Suppose A_n, C_n and Δ_n are $N(n) \times N(n)$ matrices such that*

$$\|A_n - C_n + \Delta_n\|_F^2 = o(n), \quad \mathrm{rank}\, \Delta_n = o(n).$$

Then

$$\sigma(A_n) \sim \sigma(C_n),$$

and, moreover,

$$\lambda\left(\frac{A_n + A_n^*}{2}\right) \sim \lambda\left(\frac{C_n + C_n^*}{2}\right), \quad \lambda\left(\frac{A_n - A_n^*}{2\mathrm{i}}\right) \sim \lambda\left(\frac{C_n - C_n^*}{2\mathrm{i}}\right).$$

We can reformulate the hypotheses in a more convenient form.

Theorem 2. *The hypotheses of Theorem 1 are fulfilled if and only if for any $\varepsilon > 0$ there exist matrices $\Delta_n(\varepsilon)$ such that*

$$\|A_n - C_n + \Delta_n(\varepsilon)\|_F^2 \le \varepsilon N(n), \quad \mathrm{rank}\, \Delta_n(\varepsilon) \le \varepsilon N(n)$$

for all n with sufficiently large components.

3 Preliminaries

We shall use a component-wise multiplication of two vectors:

$$h \cdot x \equiv (h_1 x_1, \ldots, h_m x_m),$$

where

$$h = (h_1, \ldots, h_m), \quad x = (x_1, \ldots, x_m).$$

Given a multiindex $n = (n_1, \ldots, n_m)$, we set

$$h_n = \left(\frac{2\pi}{n_1}, \ldots, \frac{2\pi}{n_m}\right)$$

and denote by F_n the m-level Fourier matrix

$$F_n = \left[e^{-\mathrm{i}(h_n \cdot k, l)} \right],$$

where k and l are n-admissible multiindices. Below we list some statements which are well known or easy enough to prove.

1. Let c be the first column of a multilevel circulant C_n. Then

$$C_n = \frac{1}{N(n)} F_n^* \mathrm{diag}\,(F_n c) F_n.$$

   Thus, the columns of a matrix

$$P_n = [p_k^{(n)}] = \frac{1}{\sqrt{N(n)}} F_n^*$$

   comprise a complete orthonormal set of the eigenvectors of C_n.

2. Suppose C_n is the optimal m-level circulant for the m-level Toeplitz matrix A_n. Then (using the scalar product of a Hermitian space) we have [5]

$$(A_n p_k^{(n)}, p_k^{(n)}) = (C_n p_k^{(n)}, p_k^{(n)})$$

   for any n-admissible multiindex k. The quantities in the right-hand side constitute a complete set of the eigenvalues of C_n.

3. For any vector $p = [p_k]^{\mathrm{T}}$ there holds

$$(A_n p, p) = \frac{1}{(2\pi)^m} \int\limits_{t\ \in\ \Pi^m} f(t) | \sum_k p_k \, e^{\mathrm{i}(k,t)})|^2 dt.$$

4. If $\tau = (\tau_1, \ldots, \tau_m)$ then the m-level matrices generated by functions $f(t)$ and $f(t+\tau)$ are unitarily similar:

$$A_n(f(t+\tau)) = U_n(\tau)\, A_n(f(t))\, U_n^*(\tau), \quad C_n(f(t+\tau)) = U_n(\tau)\, C_n(f(t))\, U_n^*(\tau),$$

where

$$U_n(\tau) = \text{diag}\,\{e^{\mathrm{i}\tau k}\}.$$

The first equation follows from the identity $a_k(f(t+\tau)) = e^{\mathrm{i}k\tau} a_k(f(t))$, while the second one emanates from the former and from the formulas for the elements of the optimal multilevel circulant $\{C_n\}_{kl} = \frac{1}{N(n)} \sum a_{r-s}$, where r, s run over all n-admissible multiindices subject to $r - s = k - l \, (\text{mod}\, n)$.

5. Given a sequence of splittings

$$A_n(f(t)) = A_{1n}(f(t)) + A_{2n}(f(t)), \quad C_n(f(t)) = C_{1n}(f(t)) + C_{2n}(f(t)),$$

for any $\tau = (\tau_1, \ldots, \tau)$ the matrices generated by $f(t+\tau)$ possess the similar splittings preserving the ranks and any unitarily equivalent norm of the splitting components.

6. If $p_k^{(n)}$, $k = (k_1, \ldots, k_m)$, is a column of the m-level unitary matrix P_n then

$$|\sum_k p_k^{(n)} \, e^{\mathrm{i}(k,t)})|^2 \; = \; \prod_{j=1}^m \frac{\sin^2 \left(\frac{h_{nj} k_j + t_j}{2}\right) n_j}{n_j \, \sin^2 \frac{h_{nj} k_j + t_j}{2}}, \qquad h_{nj} = \frac{2\pi}{n_j}.$$

4 Basic Lemmas

Lemma 1. *Let $0 < \delta < \pi$ and $\Pi(\delta)$ denote a cube with the side length equal to 2δ. Denote by $\mu_n(\Pi(\delta))$ the number of n-admissible multiindices k such that*

$$\max_{t \,\in\, \Pi} \; |\sum_k p_k^{(n)} \, e^{\mathrm{i}(k,t)}|^2 \; > \; \frac{c_1(\delta)}{N(n)}, \quad \textit{where} \quad c_1(\delta) = \frac{1}{\sin^{2m} \frac{\delta}{2}}.$$

Then for an arbitrary n there exists a cube $\Pi(\delta)$ such that the following inequality holds

$$\mu_n(\Pi(\delta)) \; \leq \; \delta c_2 N(n), \qquad c_2 = \frac{4m}{\pi}.$$

Proof. The statement of the lemma for the case $m = 1$ is proved in [9]. Thus, let $\mu_{n_j}(\Pi_j)$ be the number of those indices $0 \le k_j \le n_j - 1$ that satisfy the inequality

$$\max_{t_j \,\in\, \Pi_j} \frac{\sin^2\left(\frac{h_{nj}k_j + t_j}{2}\right) n_j}{n_j\, \sin^2 \frac{h_{nj}k_j + t_j}{2}} \;>\; \frac{1}{n_j\, \sin^2 \frac{\delta}{2}}.$$

Then, as is proved in [9], for any n_j it is possible to find an interval Π_j of length 2δ such that

$$\mu_{n_j}(\Pi_j) \le \delta\, \frac{4}{\pi}\, n_j.$$

It remains to verify that the choice

$$\Pi(\delta) = \Pi_1 \times \ldots \times \Pi_m$$

implies the desired estimate. □

Lemma 2. *Suppose* $f \in L_1$ *,* $f(x) \ge 0$ *and* $\operatorname{supp} f \subset \Pi(\delta) \subset [-\pi, \pi]^m$. *Then for the (Hermitian) Toeplitz m-level matrices A_n and corresponding (Hermitian) optimal m-level circulants C_n, the splittings exist*

$$A_n = A_{1n} + A_{2n}, \quad C_n = C_{1n} + C_{2n}$$

such that

$$\max\{\|A_{1n}\|_2, \|C_{1n}\|_2\} \;\le\; c_1(\delta)\|f\|_{L_1}, \quad \max\{\operatorname{rank} A_{2n}, \operatorname{rank} C_{2n}\} \;\le\; 2\delta c_2 N(n).$$

Proof. By the proposition 5 from the Preliminaries, we may assume that for any n the cube $\Pi(\delta)$ is the very one for which the estimate on $\mu_n(\Pi(\delta))$ of Lemma 1 is guaranteed. Consider a column splitting

$$P_n = [P_{1n}, P_{2n}],$$

relegating to the second submatrix the vectors $p_k^{(n)}$ with those k that are counted in $\mu_n(\Pi(\delta))$, and set

$$A_{1n} = P_n \begin{bmatrix} P_{1n}^* A_n P_{1n} & 0 \\ 0 & 0 \end{bmatrix} P_n^*, \qquad A_{2n} = P_n \begin{bmatrix} 0 & P_{1n}^* A_n P_{2n} \\ P_{2n}^* A_n P_{1n} & P_{2n}^* A_n P_{2n} \end{bmatrix} P_n^*.$$

Under the hypotheses of the theorem, every matrix A_n is Hermitian, and all its eigenvalues are nonnegative. Hence,

$$\|A_{1n}\|_2 = \lambda_{\max}(P_{1n}^* A_n P_{1n}) \le \mathrm{tr}\,(P_{1n}^* A_n P_{1n}) \le c_1(\delta)\|f\|_{L_1}.$$

When taking up the analogous splittings for the optimal m-level circulants C_n, the upper estimate on $\|C_{1n}\|_2$ is retained the same as that for $\|A_{1n}\|_2$, because, due to the proposition 2 from Preliminaries, we conclude easily that

$$\mathrm{tr}\,(P_{1n}^* C_n P_{1n}) = \mathrm{tr}\,(P_{1n}^* A_n P_{1n}). \quad \Box$$

Lemma 3. *Suppose $f \in L_1$, $f(x) \ge 0$ and the Lebesgue measure of* $\mathrm{supp}\, f$ *is equal to δ^m. Then the statements of Lemma 1 are still valid with c_1 the same and $c_2(\delta)$ dependent also on the structure of* $\mathrm{supp}\, f$.

Proof. Since the set $\mathrm{supp}\, f$ has the Lebesgue measure equal to δ, it can be covered by a denumerable (and eventually finite) set of cubes with the sum of their side lengths not greater than 2δ. For each cube we apply Lemma 1 and notice that the integral of a nonnegative function over $\mathrm{supp}\, f$ does not exceed the sum of integrals over those cubes. $\Box$

5 Distribution of Eigenvalues

Lemma 4. *Let $f \ge 0$ and $f \in L_1$. Then for an arbitrary $\varepsilon > 0$ there exist Hermitian matrices $H_n(\varepsilon)$ such that*

$$\|A_n - C_n + H_n(\varepsilon)\|_F^2 \le \varepsilon N(n), \quad \mathrm{rank}\, H_n(\varepsilon) \le \varepsilon N(n)$$

for all n with sufficiently large components.

Proof. Take $M > 0$ and consider a cut-off function

$$f_M(x) = \begin{cases} f(x), & f(x) \le M, \\ M \quad , & f(x) > M. \end{cases}$$

Set $r_M(x) = f(x) - f_M(x)$. Then

$$\lim_{M\to\infty} \|r_M\|_{L_1} = 0 \qquad \text{and} \qquad \lim_{M\to\infty} \mathrm{mes}\,\mathrm{supp}\, r_M = 0.$$

Take up any $\varepsilon > 0$ and choose $M' > 0$ such that

$$\text{mes supp}\, r_{M'} \leq \frac{\varepsilon}{2c_2}.$$

Now pick up $M = M_\varepsilon \geq M'$ such that $c_1(\varepsilon)\|r_M\|_{L_1} \leq \varepsilon$. Take into account that the inclusion $\text{supp}\, r_M \subset \text{supp}\, r_{M'}$ implies that $\text{mes supp}\, r_M \leq \frac{\varepsilon}{2c_2}$. In charm with Lemma 3, we can write

$$A_n(r_M) = A_{1n}(r_M) + A_{2n}(r_M), \quad C_n(r_M) = C_{1n}(r_M) + C_{2n}(r_M),$$

where

$$\max\{\|A_{1n}(r_M)\|_2, \|C_{1n}(r_M)\|_2\} \leq \varepsilon,$$
$$\max\{\text{rank}\, A_{2n}(r_M), \text{rank}\, C_{2n}(r_M)\} \leq \varepsilon N(n).$$

Since $f_M \in L_\infty$, we have $\|(A_n(f_M) - C_n(f_M)\|_F^2 \leq \varepsilon N(n)$ for all n with sufficiently large components [6]. It means that if $\Delta_n(\varepsilon) \equiv C_{2n}(r_M) - A_{2n}(r_M)$ then

$$\begin{aligned} ||A_n(f) - C_n(f) + \Delta_n(\varepsilon)||_F^2 &= ||(A_n(f_M) - C_n(f_M)) + (A_{1n}(r_M) - C_{1n}(r_M))\|_F^2 \\ &\leq 2||A_n(f_M) - C_n(f_M)||_F^2 + 2||A_{1n}(r_M) - C_{1n}(r_M)||_F^2 \\ &\leq (2\varepsilon + 4\varepsilon^2)N(n). \end{aligned}$$

At the same time, $\text{rank}\, \Delta_n(\varepsilon) \leq 2\varepsilon N(n)$. Since ε is arbitrary, that will do the proof. $\square$

We now abandon the assumption that $f(x)$ is of a constant sign.

Lemma 5. *If $f(x) \in \mathbb{R}$ and $f \in L_1$ then the statements of Lemma 4 remain to be valid.*

Proof. If $f(x) = f^+(x) - f^-(x)$, where $f^\pm(x) \geq 0$ and $f^\pm \in L_1$, then, obviously,

$$A_n(f) = A_n(f^+) - A_n(f^-), \quad C_n(f) = C_n(f^+) - C_n(f^-).$$

Choose an arbitrary $\varepsilon > 0$. On the strength of Lemma 3, there exist Hermitian matrices H_n^+ and H_n^- such that

$$\|A_n(f^+) - C_n(f^+) + H_n^+\|_F^2 \leq \varepsilon N(n), \quad \text{rank}\, H_n^+ \leq \varepsilon N(n);$$

$$\|A_n(f^-) - C_n(f^-) + H_n^-\|_F^2 \leq \varepsilon N(n), \quad \text{rank}\, H_n^- \leq \varepsilon N(n).$$

Consequently, if $H_n \equiv H_n^+ - H_n^-$ then we get

$$\|A_n - C_n + H_n\|_F^2 \le 4\varepsilon N(n), \quad \operatorname{rank} H_n \le 2\varepsilon N(n),$$

and this completes the proof. □

Corollary. Under the hypothese of Lemma 5, $\lambda(A_n) \sim \lambda(C_n)$.

It follows from Theorem 2.

Theorem 3. *If $f(x) \in \mathbb{R}$ and $f \in L_1$ then $\lambda(A_n) \sim f(x)$.*

Proof. By the corollary of Lemma 5, $\lambda(A_n) \sim \lambda(C_n)$. Besides, if $f \in L_1$ then $\lambda(C_n) \sim f(x)$ [6]. □

Corollary. Under the hypotheses of Theorem 3, $\sigma(A_n) \sim |f(x)|$.

It is sufficient to note that if $F(x)$ is continuous function with a finite support, then $F(|x|)$ is also such.

Theorem 4. Suppose $f(x),\ g(x) \in \mathbb{R}$ and $f,\ g \in L_1$. Then $\lambda(A_n(f+g)) \sim f(x) + g(x)$.

To prove this, we need to take into account that

$$\lambda(C_n(f+g)) \sim \lambda(C_n(f) + C_n(g));$$

apart from this we have to apply several times Lemma 3 and Theorem 2. As a matter of fact, it will be an evident modification of the proof of Theorem 3.

6 Distribution of Singular Values

Theorem 5. *Let $f(x) \in \mathbb{C}$ and $f \in L_1$. Then for any $\varepsilon > 0$ there exist matrices $\Delta_n(\varepsilon)$ such that the inequalities*

$$\|A_n - C_n + \Delta_n(\varepsilon)\|_F^2 \le \varepsilon N(n), \quad \operatorname{rank} \Delta_n(\varepsilon) \le \varepsilon N(n)$$

hold for all n with sufficiently large components.

Proof. Consider the Hermitian splittings

$$A_n = A_{1n} + \mathrm{i}A_{2n}, \quad C_n = C_{1n} + \mathrm{i}C_{2n},$$

$$A_{jn} = A_{jn}^*, \quad C_{jn} = C_{jn}^*, \quad j = 1, 2,$$

and note that C_{jn} are the optimal multilevel circulants for the Hermitian multilevel Toeplitz matrices A_{jn}. The matrices A_{1n}, C_{1n} и A_{2n}, C_{2n} are generated by the functions

$$\operatorname{Re} f(x) = (f(x) + f^*(x))/2 \quad \text{and} \quad \operatorname{Im} f(x) = -\mathrm{i}(f(x) - f^*(x))/2,$$

respectively. Since $\operatorname{Re} f(x), \operatorname{Im} f(x) \in \mathbb{R}$, it follows from Lemma 5 that for any $\varepsilon > 0$ there exist Hermitian matrices H_{jn} such that the inequalities

$$\|A_{kn} - C_{kn} + H_{kn}(\varepsilon)\|_F^2 \leq \frac{1}{4}\varepsilon N(n), \quad \operatorname{rank} H_{kn}(\varepsilon) \leq \frac{1}{2}\varepsilon N(n), \quad k = 1, 2,$$

hold for all n with sufficientlt large components. It is easy to check that the choice $\Delta_n = H_{1n} + \mathrm{i}H_{2n}$ provides us with the inequalities we are after. □

Theorem 6. *Suppose $f(x) \in \mathbb{C}$ and $f \in L_1$. Then*

$$\sigma(A_n) \sim |f(x)|, \quad \lambda\left(\frac{A_n + A_n^*}{2}\right) \sim \operatorname{Re} f(x), \quad \lambda\left(\frac{A_n - A_n^*}{2\mathrm{i}}\right) \sim \operatorname{Im} f(x).$$

Proof. The statements of the theorem are fulfilled if we replace A_n by C_n [6]. Now the distribution results under question follow immediately from Theorems 1 and 5. □

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