The electric field integral equation: theory and algorithms

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1 Introduction

About 15 years have passed since the claim made by Colton and Kress in their book [3] that the development of the integral equation methods for the direct scattering problem is close to completion. However, till present *it has not been completed* by any stretch, at least concerning the vector 3D problems for screens and nonsmooth scatterers.

When solving the monochromatic scattering problem on surfaces of arbitrary shape, one usually applies the so-called *electric field integral equation*. Its theory is given in [3] only for closed surfaces. And even for them, in the 3D case this theory is not sufficient to prove the convergence of the Galerkin method (the basic means to solve such equations numerically). As a matter of fact, for screens we had no rigorous background till 1994 [9].

It can be argued that we still have some algorithms that work quite successfully. However, we can not prove this. Moreover, we can not produce *any reliable estimate* of how accurate is the solution obtained. The splitting theory [9] is the only basis visible at present to move these questions from the standstill.

When discussing the Galerkin schemes, everybody cares first about the approximation property of the basis functions. This is yet half the matter. One should not forget to prove as well the stability property. The latter is granted for many problems, since it follows immediately from the strong ellipticity property [4, 5, 10]. But this is not the case for the vector 3D electric field equation. Due to the splitting theory we know that the principal part of the electric field operator can be split into two parts, one being "positive definite" and other "negative definite" when considered on relevant functional spaces. Consequently, the state-of-the-art algorithms should somehow match this splitting. This claim and an underlying theorem was the principal result we presented in the talk at the Oxford conference.

In this paper, we discuss *both properties*: the stability one and the approximation one.

Nevertheless, we found it worthy to begin with a detailed sketch of the splitting theory for the electric field integral equation. Then, in section 5, we discuss the stability property and prove the convergence theorem based in essence on the splitting theory.

In section 6, we turn to discussing the practical algorithms. We show that the Rao-Wilton-Glisson functions [8] possess rather poor approximation properties. At the same time, we note that these functions can be viewed as a particular case of Nedelec's functions [7] exploited successfully in finite elements for solving the Maxwell equations [1]. Once this is noted, it seems easy to devise some new sets of basis functions with better approximation estimates.

In this paper we propose some functions that keep the same simplicity as the Rao-Wilton-Glisson functions but provide much higher accuracy. This is confirmed numerically. Being of Nedelec's type, the new functions can be applied to screens and nonsmooth scatterers just as the Rao-Wilton-Glisson ones.

2 The Electric Field Integral Equation

Consider the scattering problem for monochromatic (with the $e^{-i\omega t}$ dependence upon time) incident fields E^0, H^0 on a system Ω of thin perfectly conducting screens located in the free space with the wave number k, provided that Im $k \geq 0, k \neq 0$ and all the field sources lie outside the screens. Then the scattered fields E, H can be sought in the vector potential form:

$$E = \frac{i}{k}(\mathrm{grad}\,\mathrm{div}\,(Au) + k^2 Au), \quad H = \mathrm{rot}\,(Au),$$

where

$$Au = \frac{1}{4\pi}\int\limits_{\Omega} \frac{e^{ik|x-y|}}{|x-y|}u(y)ds, \quad x \notin \bar{\Omega},$$

and $u(y)$ designates the tangential vector field on Ω.

Note that $u(y)$ can be thought about as the density of the electric current on Ω. The fields E and H defined by the above formulas satisfy the Maxwell equations in $R^3\backslash\bar{\Omega}$ and the Sommerfeld like radiation conditions at infinity.

Using the Stokes theorem and some regularity assumptions we can rewrite the expression for E as

$$E = \frac{i}{k}(\mathrm{grad}\,A(\mathrm{div}\,u) + k^2 Au), \quad x \notin \bar{\Omega},$$

where $\mathrm{div}\,u$ means the surface divergence of u and we use the same symbol A when applying the operator to vector and as well to scalar functions.

The tangential components of E (and the normal components of H) are continuous up to Ω except for the edge points. Consequently, since the tangential components of the full electric field must vanish on Ω we obtain

$$\mathrm{grad}_\tau\, A(\mathrm{div}\,u) + k^2 A_\tau u = f \equiv ik\ E^0_\tau|_\Omega, \quad x \in \Omega. \tag{2.1}$$

Equation (2.1) is known as *the electric field integral equation*.

3 Relevant Functional Spaces

For $2D$ scattering problems, the Galerkin method theory appeared simple and clear many thanks to the right choice of functional spaces in which the operator considered was both invertible and strongly elliptic (cf. [4, 5, 10]). We may not have that much in the general 3D case.

It is still possible to choose the functional spaces which secure only the Fredholm property. For surfaces of arbitrary shape, the spaces large enough to contain all solutions of physical interest were recognized and studied in [9].

Assume that

$$\Omega = \bigcup_{i=1}^{m} \Omega_i$$

is a union of disjoint screens, each Ω_i is part of a closed connected compact surface M_i, and $M_i \cap M_j = \emptyset$ for $i \neq j$. We thus have $\bar{\Omega}_i \subset M_i$ for all i and $\bar{\Omega} \subset M = \bigcup_{i=1}^{m} M_i$. Any screen boundary $\partial\Omega_i = \bar{\Omega}_i \backslash \Omega_i$ is assumed in [9] to be an infinitely smooth closed curve with no self-intersection but we dare state that much of the analysis can be extended to the case of piecewise smooth boundaries.

Consider a bounded open domain $U_\alpha \subset \mathbb{R}^2$ and denote by $H^s(U_\alpha)$ the completion of functions from $C_0^\infty(U_\alpha)$, i.e., supported on $\bar{U}_\alpha$ functions from $C^\infty(\mathbb{R}^2)$, with respect to the norm

$$||f||_s = \{ \int_{\mathbb{R}^2} (1 + |\xi|^2)^s |\hat{f}(\xi)|^2 d\xi \}^{\frac{1}{2}},$$

where $\xi = (\xi_1, \xi_2)$, $|\xi| = (|\xi_1|^2 + |\xi_2|^2)^{\frac{1}{2}}$ and $\hat{f}(\xi)$ is the integral Fourier transform of $f(x)$, $x = (x_1, x_2)$. If functions f are vector-valued, say, $f = (f^1, f^2)$, then we use the same notation $H^s(U_\alpha)$ for those functions whose scalar components f^1 and f^2 belong to the previously defined $H^s(U_\alpha)$. In this case we adopt the definition

$$||f||_s = (||f^1||_s^2 + ||f^2||_s^2)^{1/2}.$$

We want to consider functions defined on M. To this end we take up a one-to-one mapping $\gamma_\alpha : U_\alpha \rightarrow \mathbb{R}^3$ which yields a bounded open domain on M. Thus, any function $f \in H^s(U_\alpha)$ can be considered as a function defined on $\gamma_\alpha(U_\alpha) \subset M$. We suppose that domains $\gamma_\alpha(U_\alpha)$, $\alpha = 1, \ldots, N$, make up a finite covering of M. Let $\{\varphi_\alpha\}_{\alpha=1}^{N}$ be a partition of unity subordinate to this covering. Then any function f defined on M can be written in the form

$$f = \sum_{\alpha=1}^{N} f_\alpha,$$

where $f_\alpha \equiv \varphi_\alpha \, f$ can be considered as a function defined on $U_\alpha \subset \mathbb{R}^2$. If $f \in C^\infty(M)$, then $f_\alpha \in C_0^\infty \, (U_\alpha)$. We now define the Sobolev space $H^s(M)$ as the completion of $C^\infty(M)$ with respect to the norm

$$||f||_s = \left(\sum_{\alpha=1}^{N} ||f_\alpha||_s^2 \right)^{1/2} .$$

As previously we use the same notation $H^s(M)$ for scalar and vector-valued functions. We also introduce two more spaces as follows:

$$\begin{aligned} H^s(\Omega) &\equiv \{f\,|_\Omega \,:\, f \in H^s(M)\}, \\ \tilde{H}^s(\Omega) &\equiv \{f \in H^s(M) \,:\, \text{supp}\ f \subset \bar{\Omega}\}. \end{aligned}$$

The last space can be viewed as the completion of $C_0^\infty(\Omega)$ with respect to the same norm $||\,.\,||_s$. Note that scalar products on $H^s(M)$ and on $\tilde{H}^s(\Omega)$ are defined in a natural way. The norm on $H^s(\Omega)$ is introduced as

$$||g||_{H^s(\Omega)} \equiv \inf\{||f||_s \,:\, f \in H^s(M),\ f|_\Omega = g\}.$$

Since the parallelogram identity is fulfilled for this norm we can easily use it to construct the scalar product on $H^s(\Omega)$.

Consider the operator from (2.1):

$$\mathcal{A}\, u = \text{grad}_\tau\ A\,(\text{div}\, u) + k^2 A_\tau u.$$

It will become clear later that $\mathcal{A}$ can be applied to any function u such that $u \in \tilde{H}_{-1/2}(\Omega)$ and $div\ u \in \tilde{H}_{-1/2}(\Omega)$ and this suggests how to define the space of origins. Specifically, the space $W = W(\Omega)$ is introduced as the completion of $C_0^\infty(\Omega)$ with respect to the norm

$$||u||_W \equiv (||u||^2_{-1/2} + ||div\ u||^2_{-1/2})^{1/2}.$$

This is the Hilbert space with the scalar product

$$(u, v)_W \equiv (u, v)_{-1/2} + (\text{div}\, u, \text{div}\, v)_{-1/2}.$$

It can be proved [9] that

$$W = \{u \in \tilde{H}^{-1/2}(\Omega) \quad : \ \text{div}\ \ u \in \tilde{H}^{-1/2}(\Omega)\},$$

where the operation $\text{div}\, u$ is extended over all $u \in \tilde{H}^{-1/2}(\Omega)$ with the help of the Fourier transform. It can be also proved [9] that W is located strictly in between $\tilde{H}^{1/2}(\Omega)$ and $\tilde{H}^{-1/2}(\Omega)$:

$$\tilde{H}^{1/2}(\Omega) \subset W \subset \tilde{H}^{-1/2}(\Omega),$$

and the inclusion mappings are continuous.

The operator $\mathcal{A}$ transforms the origins from W into images which bear the meaning of the tangential electric field components. The Maxwell equations state that the magnetic field is proportional to $\operatorname{rot} E$ and therefore the energy boundedness considerations suggest that E and $\operatorname{rot} E$ should be square-integrable over any bounded region of $\mathbb{R}^3$. We may thus anticipate that some "restrictions" of E and $\operatorname{rot} E$ on M should belong to $H^{-1/2}(M)$. The role of these "restrictions" will be played by the tangential component of E and the normal component of $\operatorname{rot} E$.

Thus, the space $W' = W'(\Omega)$ of images will be defined as

$$W' = \{ \, f|_\Omega \quad : \quad f \in H^{-1/2}(M), \ \operatorname{rot}_\nu f \in H^{-1/2}(M)\},$$

where $\operatorname{rot}_\nu f$ is the "surface rotor" (the normal component of $\operatorname{rot} f$, the vector field f being the extension of the surface vector field over a neighborhood of M; it is easy to see that any extension produces the same value of $\operatorname{rot}_\nu f$).

It can be proved that W' is antidual to W with respect to the L^2 scalar product sesquilinear form. This means that (similar to the Riesz theorem) any bounded linear functional $f(u)$ on W can be written in the form

$$f(u) = \int\limits_\Omega u\bar{v} \; ds \equiv (u, \ v)$$

for some uniquely determined $v \in W'$. We have

$$H^{1/2}(\Omega) \subset W' \subset H^{-1/2}(\Omega),$$

and $\mathcal{A}\, u \in W'$ for all $u \in W$ [9].

However, we have not shown yet that $\mathcal{A}$ is correctly defined on all of W. Using vector analysis formulas we have

$$(\mathcal{A}\, u, \ v) = -(A(\operatorname{div} u), \ \operatorname{div} v) + k^2 \ (A_\tau u, \ v) \quad \text{for any} \quad u, \ v \in C_0^\infty(\Omega).$$

Since $C_0^\infty(\Omega)$ is dense in W, it is sufficient to verify that the sesquilinear form $(\mathcal{A}\, u, \ v)$ is bounded on W and then we can define the action of $\mathcal{A}$ on an arbitrary $u \in W$ by the continuity property.

It is sufficient to prove that the forms $(Au, \ v)$ and $(A_\tau u, \ v)$ are bounded on W, which is a relatively simple task because this time we are to deal with the well-familiar weakly singular integral operator A applied to scalar functions. Anyway we know (see [9, 10]) that the operators

$$A \ : \ \tilde{H}^{-1/2}(\Omega) \ \rightarrow \ H^{1/2}(\Omega),$$

$$A_\tau \ : \ \tilde{H}^{-1/2}(\Omega) \ \rightarrow \ H^{1/2}(\Omega)$$

are bounded Fredholm operators with index zero. It remains to note that the form $(w, \ v)$ is bounded when $w \in H^{1/2}(\Omega)$ and $v \in \tilde{H}^{-1/2}(\Omega)$.

We thus proved that the form $(\mathcal{A}\, u, \ v)$ is indeed bounded on W and hence for any $u \in W$ the element Au can be identified with a bounded linear functional on W, which in its turn can be uniquely represented by some element from W'.

The equality in (2.1) should be understood as the equality of two elements from W'. Equivalently, we can call $u \in W$ a (generalized) solution to the equation (2.1) if [9]

$$-\,(A\ (\mathrm{div}\, u), \mathrm{div}\ \ v) + k^2\ (A_\tau u,\ v) = (f,\ v) \quad \text{for all} \quad v \in C_0^\infty. \tag{3.1}$$

4 The Operator Splitting

Any sufficiently smooth vector field supported on a bounded region in $\mathbb{R}^3$ can be uniquely expanded into the divergence-free and the rotor-free (potential) components. The same applies to vector fields defined on manifolds in $\mathbb{R}^3$ [9]. Denote by W_1 and W_2 the completions of $W_1^0 = \{\ u \in C_0^\infty\ (\Omega)\ :\ \mathrm{div}\ \ u = 0\}$ and $W_2^0 = \{\ u \in C_0^\infty\ (\Omega)\ :\ u = \mathrm{grad}\ \ h,\ \ h \in C_0^\infty\ (\Omega)\}$, respectively, with respect to the W-norm. Then W_1 and W_2 are closed linear manifolds, and W is their direct sum:

$$W = W_1 \oplus W_2.$$

Analogously, we have

$$W' = W_1' \oplus W_2',$$

where

$$W_1' = \{\ f \in W'\ :\ \mathrm{div}\, f = 0\} \quad \text{and} \quad W_2' = \{\ f \in W'\ :\ \mathrm{rot}_\nu\, f = 0\}.$$

The subspace W_1' is antidual to W_1 and W_2' is antidual to W_2. The projection operator on W_1 in parallel to W_2 is bounded in W while the projection operator on W_1' in parallel to W_2' is bounded in W'. Moreover, the W-norm on W_1 is equivalent to the $\tilde{H}^{-1/2}(\Omega)$-norm while the W-norm on W_2 is equivalent to the $\tilde{H}^{1/2}(\Omega)$-norm, and, similarly, the W'-norm on W_1' is equivalent to the $H^{1/2}(\Omega)$-norm while the W'-norm on W_2' is equivalent to the $H^{-1/2}(\Omega)$-norm. All these facts were proved in [9].

The operator $\mathcal{A}$ can be split quite naturally:

$$\mathcal{A} = A^{(1)} + k^2\ A^{(2)},$$

where

$$A^{(1)}\ u = \mathrm{grad}_\tau\ \ \hat{A}\ (\mathrm{div}\ \ u), \quad A^{(2)}\ u = \hat{A}_\tau\ u.$$

Allowing for $W = W1 \oplus W2$ and $W' = W'1 \oplus W'2$, we will benefit by representing the operators in the 2×2 matrix form

$$A^{(1)} = \begin{bmatrix} A_{11}^{(1)} & A_{12}^{(1)} \\ A_{21}^{(1)} & A_{22}^{(1)} \end{bmatrix}, \quad A^{(2)} = \begin{bmatrix} A_{11}^{(2)} & A_{12}^{(2)} \\ A_{21}^{(2)} & A_{22}^{(2)} \end{bmatrix},$$

where $A_{ij}^{(k)}\ :\ W_j\ \rightarrow\ W_i'$ are bounded linear operators $(i,\ j,\ k = 1, 2)$.

It is easy to verify that

$$A^{(1)} = \begin{bmatrix} 0 & 0 \\ 0 & A_{22}^{(1)} \end{bmatrix}.$$

Somewhat more difficult is to prove that $A_{22}^{(1)} : W_2 \to W_2'$ can be split into $A_{22}^{(1)} = -A_1^{(1)} + A_2^{(1)}$, where $A_1^{(1)} : W_2 \to W_2'$ is a strongly elliptic and hence continuously invertible operator while $A_2^{(1)}$ is a compact operator [9]. From the theory of pseudodifferential operators,

$$A = A_1 + A_2 + A_3,$$

where A_1 is a pseudodifferential operator of order -1, A_2 is a pseudodifferential operator of order -2, and A_3 is an operator with the infinitely smooth kernel. Consequently, we can set

$$A_1^{(1)} u = -\mathrm{grad}\, A_1\ (\mathrm{div}\, u), \quad A_2^{(2)} u = \mathrm{grad}\,(A_2 + A_3)\ (\mathrm{div}\, u), \qquad u \in W_2.$$

Since $\mathrm{div}\, u \in \tilde{H}^{-1/2}(\Omega)$ we have $(A_2+A_3)(\mathrm{div}\, u) \in H^{3/2}(\Omega)$, and finally $A_2^{(2)} u \in H^{1/2}(\Omega)$. Taking into account that the inclusion mapping $H^{1/2}(\Omega) \subset H^{-1/2}(\Omega)$ is compact and recollecting that the W'-norm on W_2' is equivalent to the $H^{-1/2}(\Omega)$-norm, we conclude that $A_2^{(2)}$ is a compact operator.

Similar arguments permit us to prove that the operators $A_{12}^{(2)}$, $A_{21}^{(2)}$, and $A_{22}^{(2)}$ are compact and that $A_{11}^{(2)} = A_1^{(2)} + A_2^{(2)}$, where $A_1^{(2)} : W_1 \to W_1'$ is a strongly elliptic and hence continuously invertible operator while $A_2^{(2)}$ is a compact operator.

Combining all the above operator decompositions we can formulate the following key-note theorem [9].

Theorem 1. *The operator* $\mathcal{A} : W \to W'$ *can be split into*

$$\mathcal{A} = D + C,$$

where C *is a compact operator while* D *allows for the diagonal matrix representation*

$$D = \begin{bmatrix} k^2\ D_1 & 0 \\ 0 & -D_2 \end{bmatrix}$$

with the strongly elliptic operators $D_1 : W_2 \to W_2'$ *and* $D_2 : W_1 \to W_1'$.

Clearly, for $k \neq 0$ the operator $\mathcal{A}$ is a Frefholm operator with the zero index. If $\mathrm{Im}\, k \geq 0$, $k \neq 0$ and all the screens are open then $\mathcal{A}$ is in fact a continuously invertible operator from W to W' [9].

5 The Galerkin Method in the Vector 3D Case

The expounded above theory lays a firm ground for work with the electric field integral equation $\mathcal{A}\, u = f$. Since $\mathcal{A}$ possesses the Fredholm property, for non-resonant wave number values we may assume that $\mathcal{A}$ is a continuously invertible operator. This is automatically fulfilled if all the screens are open.

Consider an n-dimensional space $V_n \subset W$ and let us approximate u by an element $u_n \in V_n$. The Galerkin method suggests that u_n is sought from the Galerkin equations

$$(\mathcal{A}\, u_n, v) = (f, v) \quad \forall\, v \in V_n. \tag{5.1}$$

Obviously, these equations define a finite dimensional operator $\mathcal{A}_n$; $V_n \to V_n'$, where V_n' is antidual to V_n.

The convergence property ($u_n \to u$ as $n \to \infty$) follows immediately from the following two assumptions.

The approximation property. For any $v \in W$ there exist $v_n \in V_n$ such that $v_n \to v\ for\ n \to \infty$.

The stability property. There exists $c > 0$ such that for n sufficiently large, $||\mathcal{A}_n u||_{W'} \geq c||u||_W$ for all $u \in V_n$ uniformly in n.

If these two properties are fulfilled then

$$||u_n - u||_W \leq C \inf_{v \in V_n} ||v - u||_W,$$

which is considered usually as the quasioptimal convergence. The derivation of the convergence rate estimates reduces thus to the standard problem of the approximation theory.

The stability property is guaranteed if $\mathcal{A}$ is strongly elliptic, i.e.,

$$(\mathcal{A}\, v, v) \geq c(v, v)\ \forall\, v \in W,$$

or, in a generalized form, if $\mathcal{A}$ satisfies the Gårding inequality

$$\text{Re}\ ((A + K)v, v) \geq c||v||_W\ \forall\, v \in W$$

for some compact operator K [5].

The principal difficulty with the electric field equation consists in that $\mathcal{A}$ is no longer strongly elliptic. This implies immediately that there exists a sequence of subspaces $V_n \subset W$ such that the solutions u_n to the Galerkin equations do not converge to u. Moreover, it can be shown that such spaces V_n can be spanned by truncations up to nth element of some orthogonal basis of W [4].

However, we are now able to propose some Galerkin schemes which are guaranteed to converge.

Theorem 2. *Let the electric field integral equation be such that the operator* $\mathcal{A}$ *is invertible, and assume that n-dimensional subspaces*

$$V_n^1 \subset W_1 \quad \text{and} \quad V_n^2 \subset W_2$$

possess the approximation property in W_1 *and* W_2*, respectively. Then the Galerkin method on subspaces* $V_n \equiv V_n^1 + V_n^2$ *is guaranteed to converge.*

Proof. Consider the splitting $\mathcal{A} = D + C$, where D is the continuously invertible operator and C is the compact operator. When applied to the equation $Du = f$, the Galerkin method under question reduces to solving simultaneously the pair of equations $k^2 D_1 u^1 = f^1$, $-D_2 u^2 = f^2$, where $f = f^1 + f^2$, $f^1 \in W_1'$ and $f^2 \in W_2'$.

Since the operators D_1 and D_2 are strongly elliptic, the Galerkin method converges for $Du = f$. From the theory of projection method [4] it follows that the Galerkin method converges also for any invertible operator of the form $D+K$, where K is a compact operator. Setting $K = C$ we complete the proof.

It should be emphasized that even for the operator D the standard Galerkin method with V_n that do not match the splitting of theorem 1 may not converge.

6 Practical Numerical Algorithms

Practical solution of the electric field integral equation on a surface Ω begins with approximating Ω by a polyhedron Ω_h, very often with triangular panels (let h denote their maximal diameter).

Similarly, when solving the Maxwell equations in the 3D case, one uses finite elements supported on tetrahedrons. Nedelec's elements conforming in $H(\text{div}\,)$ seem to be a good choice in that case. For the surface equation (2.1), the solution space W can be viewed as $H^{-1/2}(\text{div}\,)$. The Rao-Wilton-Glisson functions [8] are thus the "surface finite elements" conforming in W.

Consider a triangular element T with vertices T_i, $i = 1, 2, 3$. All that pertains to the side opposite to T_i will be marked with the index i. Let l_i and h_i denote the length of the corresponding side and height dropped on it from T_i. Let n_i be the outward normal vector to the ith side.

For any point $(x, y) \in T$ let $\xi_i = \xi_i(x, y)$ denote the ratio of the distance between (x, y) and T_i to h_i. Thus, ξ_i, 1, 2, 3, are the so-called barycentric coordinates of (x, y) (they are interdependent, for $\xi_1 + \xi_2 + \xi_3 = 1$ for any $(x, y) \in T$).

Consider a piecewise continuous vector field v on a surface, assuming that all discontinuities belong to boundaries of curvilinear triangles. Then $v \in W$ ($v \in W'$) is guaranteed whenever the normal component (the tangential component) of v is continuous along these boundaries. Thus, the "surface finite element" on T will be conforming in W if its degrees of freedom provide the continuity of the normal component.

To this end, it is convenient to construct the *edge elements* on T, i.e., the vector functions on T whose normal component is zero on all the edges except for one (cf. [1, 7]).

Consider a vector field f and its polynomial approximation

$$p = \begin{bmatrix} p_1(x,y) \\ p_2(x,y) \end{bmatrix}$$

on T. Let p enjoy the following properties:

$$(p(T_i), n_j) = f_{ij} \equiv (f(T_i), n_j), \quad j \neq i.$$

It can be proved that p does indeed if it is of the form

$$p = \sum_{i=1}^{3} \sum_{\substack{j=1^3 \\ j \neq i}} f_{ij} \sigma_{ij} e_{ij},$$

where e_{ij} are the very edge elements:

$$e_{ij} = l_j \xi_i \nabla' \xi_k, \quad i \neq j,\ k \neq i,\ k \neq j;$$

$$\nabla' \xi \equiv \begin{bmatrix} \frac{\partial \xi}{\partial y} \\ -\frac{\partial \xi}{\partial x} \end{bmatrix};$$

$$\sigma_{ij} = \begin{cases} -1, & \text{if} \quad (i,j) \in \{(1,2), (2,3), (3,1)\}, \\ 1, & \text{if} \quad (i,j) \in \{(1,3), (2,1), (3,2)\}, \\ 0, & \text{otherwise.} \end{cases}$$

Choosing $f_{ij} = c_j$ we have 3 degrees of freedom (c_1, c_2, and c_3). It is easy to verify that the functions

$$e_{13} - e_{23}, \quad e_{32} - e_{12}, \quad e_{21} - e_{31}$$

are the familiar Rao-Wilton-Glisson functions on T.

These functions can be regarded as Nedelec-like elements of the zero order. Using the Bramble-Hilbert lemma [2] we can state that the Rao-Wilton-Glisson functions approximate functions from H^1 with the $O(h^{1/2})$ accuracy in the W-norm.

We can be better off with maintaining all 6 degrees of freedom (the basis functions on T will be the above $\sigma_{ij} e_{ij}$, $i \neq j$). In this case, the same Bramble-Hilbert lemma implies that functions from H^2 are approximated with the $O(h^{3/2})$ accuracy.

One may speculate about twice the number of unknowns for the same number of triangular patches. However, the error in the solution is expected to fall down more than twice.

This was confirmed numerically. We considered the scattering of the plane wave with $k = 1$ on the sphere of radius 1. The comparison of our method with the Rao-Wilton-Glisson one is given in the following table:

The Rao-Wilton-Glisson Method

The number of unknowns	72	162	288	450
The relative L_2-error	0.151	0.108	0.100	0.098

Our Method

The number of unknowns	144	324	576	
The relative L_2-error	0.100	0.048	0.028	

The choice of the L_2-norm is typical for practice. The W-norm of the above theory is of course different but too difficult to calculate.

Both methods (the Rao-Wilton-Glisson and ours) do not match the operator splitting of theorem 1. However, the method with 6 degrees of freedom can be easily modified to match it yet in part. Instead of functions $\sigma_{ij} e_{ij}$ we can take up their linear combinations

$$e_{13} \pm e_{23}, \quad e_{32} \pm e_{12}, \quad e_{21} \pm e_{31}.$$

It is easy to check that the functions with the plus sign are divergence-free. The functions with the minus sign are the familiar Rao-Wilton-Glisson functions (which are not rotor-free). Such a version of basis functions does not provide any additional gain in accuracy. All the same, it can be useful because it gives naturally an idea of some practical procedure of the error estimation.

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