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Structured Preconditioners for Some Operator Equations

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SUMMARY

We propose and report on the use of block circulant preconditioners for solution of non-Hermitian linear systems in one important typical application from computational electromagnetics (evaluation of fields from localized sources in a heterogeneous isotropic formation uniform in one particular direction). The essentials of the efficiency and discussion are the fast approximation method (based on Fourier polynomials) and fast algebraic solver (GMRES with block circulant preconditioners). Copyright © 2002 John Wiley & Sons, Ltd.

KEY WORDS: Block circulant preconditioners; integral equations; computational electromagnetics

1. Introduction

In this paper we report on the use of block circulant preconditioners for solution of non-Hermitian linear systems arising in the evaluation of electromagnetic fields from localized sources in a heterogeneous isotropic formation with conductivity depending only upon x and y coordinates and independent of z. Such problems are simpler than general 3D but far more complicated than 2D ones, because of the source. By the integral Fourier transform they can be reduced to infinitely many 2D problems and thence are usually referred to as 2.5D problems. The main difficulties are anyway concerned with the involved 2D problems. Each of them can be reduced to a system of two Helmholtz equations with respect to the z-components of the Fourier images of the electric and magnetic fields [1].

The interface boundaries are closed cylinders parallel to the z axis and become closed contours in the x and y coordinates of the 2D problems. In simulation, the electric parameters are often assumed to be piecewise constant [1]. Using single-layer potentials, we come up with systems of integral equations. Then, we approximate the potentials by Fourier polynomials and obtain the algebraic equations by collocation.

As we found, this particular application is one where the method of integral equations can be much faster (1000 times and more, in some cases) and a lot more accurate than solution

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procedures based directly on the partial differential formulation. The essentials of the efficiency and discussion below are the following:

- *fast approximation method* (based on the Fourier polynomials);
- *fast algebraic solver* (based on the block circulant preconditioners).

2. Mathematical model and integral equations

Let the electric and magnetic fields be sought in the form of Fourier integrals

$$\boldsymbol{E} = \frac{1}{2\pi}\int\limits_{-\infty}^{\infty} E(x,y,\zeta)\mathrm{e}^{\mathrm{i}\zeta z}d\zeta, \quad \boldsymbol{H} = \frac{1}{2\pi}\int\limits_{-\infty}^{\infty} H(x,y,\zeta)\mathrm{e}^{\mathrm{i}\zeta z}d\zeta.$$

Then, the homogeneous Maxwell equations [†]

$$\mathrm{rot}\,\boldsymbol{E} = \mathrm{i}\omega\mu\boldsymbol{H}, \quad \mathrm{rot}\,\boldsymbol{H} = -\mathrm{i}\omega\varepsilon\boldsymbol{E}$$

are equivalent to infinitely many independent 2D equations with respect to $E(x,y,\zeta)$ and $H(x,y,\zeta)$, where ζ is the Fourier parameter. These 2D equations are of the following component-wise form:

$$\begin{aligned} \frac{\partial E_z}{\partial x} &= \mathrm{i}\zeta E_x - \mathrm{i}\omega\mu H_y, & \frac{\partial E_z}{\partial y} &= \mathrm{i}\zeta E_y + \mathrm{i}\omega\mu H_x, \\ \frac{\partial H_z}{\partial y} &= -\mathrm{i}\omega\varepsilon E_x + \mathrm{i}\zeta H_y, & \frac{\partial H_z}{\partial x} &= \mathrm{i}\omega\varepsilon E_y + \mathrm{i}\zeta H_x, \end{aligned}$$

$$\begin{aligned} \frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} &= \mathrm{i}\omega\mu H_z, \\ \frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} &= -\mathrm{i}\omega\varepsilon E_z. \end{aligned}$$

Set $k^2 = \omega^2\varepsilon\mu - \zeta^2$. All the quantities can be expressed through two of them, E_z and H_z, as follows [1]:

$$\begin{aligned} E_x &= \frac{1}{k^2}(\mathrm{i}\zeta\frac{\partial E_z}{\partial x} + \mathrm{i}\omega\mu\frac{\partial H_z}{\partial y}), & H_x &= \frac{1}{k^2}(\mathrm{i}\zeta\frac{\partial H_z}{\partial x} - \mathrm{i}\omega\varepsilon\frac{\partial E_z}{\partial y}), \\ E_y &= \frac{1}{k^2}(\mathrm{i}\zeta\frac{\partial E_z}{\partial y} - \mathrm{i}\omega\mu\frac{\partial H_z}{\partial x}), & H_y &= \frac{1}{k^2}(\mathrm{i}\zeta\frac{\partial H_z}{\partial y} + \mathrm{i}\omega\varepsilon\frac{\partial E_z}{\partial x}). \end{aligned}$$

[†] All the fields depend on time harmonically with the given frequency ω; the equations hold in domains where μ and ε are continuous, additional interface conditions are imposed on the boundaries between these domains. We assume that μ and ε are piece-wise constant.

Now suppose that the formation consists of several nested domains with smooth boundaries. Let the resistivity be constant in any piece in between of the contours. Denote the above contours by $\Gamma_1, \ldots, \Gamma_m$. We assume that the curves do not intersect and Γ_1 is the innermost one embracing the only domain with source of the electromagnetic fields. Thus, there are $m+1$ subdomains $\Omega_1, \ldots, \Omega_{m+1}$ separated by these curves. The source is located in Ω_1.

Let E_z^0, H_z^0 be the primary Fourier field defined as solution to the Fourier-transformed Maxwell equations in the uniform space with electric parameters of Ω_1. Then we introduce the shifted Fourier fields as follows:

$$E_z^s = \begin{cases} E_z - E_z^0 & \text{in } \Omega_1, \\ E_z, & \text{otherwise;} \end{cases} \qquad H_z^s = \begin{cases} H_z - H_z^0 & \text{in } \Omega_1, \\ H_z, & \text{otherwise.} \end{cases}$$

Inside any subdomain E_z^s and H_z^s satisfy the following independent Helmholtz equations:

$$\frac{\partial^2}{\partial x^2} E_z^s + \frac{\partial^2}{\partial y^2} E_z^s + k^2 E_z^s = 0, \tag{1}$$

$$\frac{\partial^2}{\partial x^2} H_z^s + \frac{\partial^2}{\partial y^2} H_z^s + k^2 H_z^s = 0, \tag{2}$$

where

$$k = k(\Omega_i), \quad i = 1, \ldots, m+1.$$

The continuity of tangential components across the boundaries can be expressed by the four equations as follows:

$$[E_z^s] = 0, \quad [H_z^s] = 0, \tag{3}$$

$$\left[\frac{\zeta}{k^2}\frac{\partial E_z^s}{\partial \tau} - \frac{\omega\mu}{k^2}\frac{\partial H_z^s}{\partial n}\right] = 0, \quad \left[\frac{\zeta}{k^2}\frac{\partial H_z^s}{\partial \tau} + \frac{\omega\varepsilon}{k^2}\frac{\partial E_z^s}{\partial n}\right] = 0. \tag{4}$$

Since E_z^s and H_z^s are solutions to the Helmholtz equations, they can be obtained as *single-layer potentials*:

$$E_z^s(M_0) = \frac{\mathbf{i}}{4}\int\limits_{\Gamma_{i-1}} H_0^{(1)}(kr)\varphi_{i-1}^+(M)ds(M) + \frac{\mathbf{i}}{4}\int\limits_{\Gamma_i} H_0^{(1)}(kr)\varphi_i^-(M)ds(M), \tag{5}$$

$$H_z^s(M_0) = \frac{\mathbf{i}}{4}\int\limits_{\Gamma_{i-1}} H_0^{(1)}(kr)\psi_{i-1}^+(M)ds(M) + \frac{\mathbf{i}}{4}\int\limits_{\Gamma_i} H_0^{(1)}(kr)\psi_i^-(M)ds(M), \tag{6}$$

$$i = 1, \ldots, m+1.$$

Here, r is the distance between the points M_0 and M and $H_0^{(1)}$ is the Hankel function of the first kind of order zero. ‡ For unification, it is assumed that $\Gamma_0 = \Gamma_{m+1} = \emptyset$. "Plus" stands for the subdomain external to the corresponding contour, "minus" is for the internal subdomain.

There are $4m$ unknown density functions $\varphi_i^{\pm}$, $\psi_i^{\pm}$, $i = 1, \ldots, m$. They can be found from the equations (3) and (4) imposed on each boundary. Since E_z^s and H_z^s are sought in the form (5) and (6), these equations are obviously a system of $4m$ integral equations.

‡ $H_0^{(1)}$ is the fundamental solution to (1) and (2) satisfying the radiation conditions at infinity. Consequently, the potentials (5) and (6) satisfy (1) and (2).

3. Fast approximation method

Assume that all functions are 2π-periodic and consider integrals of the form

$$\psi(\tau) = \int_0^{2\pi} g(\tau,t)\varphi(t)dt,$$

where $\varphi(t)$ is a sufficiently smooth 2π-periodic function. The kernel $g(\tau,t)$ is smooth for all $t \neq \tau$ but can be infinite at $t = \tau$. More precisely, we always have

$$g(\tau,t) = g_1(\tau,t) + g_2(\tau,t)$$

with a special-form function g_1, and g_2 smooth and 2π-periodic in both variables.

Three basic prototypes for g_1 are the following:

$$\begin{aligned} L(\tau,t) &= -\frac{1}{2\pi}\ln\left|\sin\left(\frac{\tau-t}{2}\right)\right|, \\ S(\tau,t) &= -\frac{1}{4\pi}\cot\left(\frac{\tau-t}{2}\right), \\ H(\tau,t) &= -\frac{1}{8\pi}\sin^{-2}\left(\frac{\tau-t}{2}\right). \end{aligned}$$

These are kernels of the logarithmic (Symm), singular (Cauchy) and hypersingular (Hadamard) integral operators

$$\mathcal{L}\varphi = \int_0^{2\pi} L(\tau,t)\varphi(t)dt, \quad \mathcal{S}\varphi = \int_0^{2\pi} S(\tau,t)\varphi(t)dt, \quad \mathcal{H}\varphi = \int_0^{2\pi} H(\tau,t)\varphi(t)dt.$$

Note that

$$\mathcal{S} = \frac{\partial}{\partial\tau}\mathcal{L}, \quad \mathcal{H} = -\frac{\partial}{\partial\tau}\mathcal{S}.$$

We can evaluate $\mathcal{L}$, $\mathcal{S}$ and $\mathcal{H}$ using the following spectral properties [5]:

$$\mathcal{L}(\mathrm{e}^{\mathrm{i}nt}) = \begin{cases} \frac{1}{2|n|}\,\mathrm{e}^{\mathrm{i}n\tau}, & n \neq 0, \\ \ln 2, & n = 0; \end{cases}$$

$$\mathcal{S}(\mathrm{e}^{\mathrm{i}nt}) = \begin{cases} \frac{\mathrm{i}\,n}{2|n|}\,\mathrm{e}^{\mathrm{i}n\tau}, & n \neq 0, \\ 0, & n = 0; \end{cases}$$

$$\mathcal{H}(\mathrm{e}^{\mathrm{i}nt}) = \begin{cases} \frac{|n|}{2}\,\mathrm{e}^{\mathrm{i}n\tau}, & n \neq 0, \\ 0, & n = 0. \end{cases}$$

Given N values of the function φ on a uniform grid, we compute the values of ψ at the same

nodes. For definiteness, let

$$\varphi_j = \varphi(t_j), \quad t_j = hj, \quad j = 0, \ldots, N-1; \qquad h = 2\pi/N.$$

Basic Singularity Case:

- Interpolate $\varphi(t)$ by a trigonometric polynomial
$$P_N(t) = \sum_{-N/2<n\le N/2} a_n \mathrm{e}^{\mathbf{i}nt}$$
on the values $\varphi_j = \varphi(t_j),\ j = 0, \ldots, N-1$.
- Apply the spectral properties to obtain the image Q_N of P_N under action of $\mathcal{L}$, $\mathcal{S}$, or $\mathcal{H}$ (this is a trigonometric polynomial of the same order).
- Take $\psi_j = Q_N(t_j)$ as approximate values of $\psi(t_j)$.

General Kernel Case:

- Given a general kernel $g(\tau, t)$, find g_1 of the form
$$g_1 = \alpha H + \beta S + \gamma L$$
with
$$\alpha = \sum a_i(\tau)\hat{a}_i(t), \quad \beta = \sum b_i(\tau)\hat{b}_i(t), \quad \gamma = \sum c_i(\tau)\hat{c}_i(t),$$
and reduce the integration of g_1 to several applications of the above basic singularity procedure with appropriately modified φ.
- For integration of the smooth constituent g_2 use the formula of rectangles on the same uniform grid.

Formally the separable functions α, β and γ may involve arbitrarily many members and are not uniquely determined. One should be careful because some of the equivalent representations of α, β and γ can lead to a loss of accuracy.

The accuracy depends on how well $\hat{a}_i(t)$, $\hat{b}_i(t)$, $\hat{c}_i(t)$ and g_2 can be approximated by trigonometric polynomials. This is obviously linked with the smoothness properties. In our application α, β and γ have only one term.

4. Fast algebraic solver

In the case of m nested domains, the collocation method along with the above approximation scheme yields a system of linear algebraic equations with $4mn$ unknowns so long as n nodes are taken on each boundary. The nest-structure of contours is reflected in the following block structure of the algebraic system:

$$\begin{bmatrix} A_1 & C_1 & & \\ B_2 & A_2 & C_2 & \\ \ddots & \ddots & \ddots & \\ & B_{m-1} & A_{m-1} & C_{m-1} \\ & & B_m & A_m \end{bmatrix} \begin{bmatrix} U_1 \\ U_2 \\ \cdots \\ U_{m-1} \\ U_m \end{bmatrix} = \begin{bmatrix} F_1 \\ 0 \\ \cdots \\ 0 \\ 0 \end{bmatrix}.$$

Denote this system by $Au = f$. To solve it, we can apply a direct method based on the LU decomposition. The complexity will be $O(mn^3)$ due to the block tridiagonal structure (and $O(m^3n^3)$ if this structure is ignored). The matrix-vector multiplication complexity for this matrix is $O(mn^2)$.

In spite of a quite agreeable structure of A, iterations are still a keen alternative to the direct solvers. We apply GMRES with no limits on the Krylov subspace dimension. However, iterations can lag if no preconditioner is used. One preconditioner of choice is a *direct block circulant preconditioner* arising as the coefficient matrix for the concentric circles. In that case, all the matrices A_i, B_i, C_i are of the form

$$G = \begin{bmatrix} G_{11} & G_{12} & G_{13} & G_{14} \\ G_{21} & G_{22} & G_{23} & G_{24} \\ G_{31} & G_{32} & G_{33} & G_{34} \\ G_{41} & G_{42} & G_{43} & G_{44} \end{bmatrix},$$

where each block G_{ij} is a circulant matrix of order n. The latter means that $C = G_{ij}$ is of the following very special form:

$$C = \begin{bmatrix} c_0 & c_{n-1} & \dots & \dots & c_2 & c_1 \\ c_1 & c_0 & c_{n-1} & \dots & \dots & c_2 \\ \dots & \ddots & \ddots & \ddots & \ddots & \dots \\ \dots & \ddots & \ddots & \ddots & \ddots & \dots \\ c_{n-2} & \ddots & \ddots & c_1 & c_0 & c_{n-1} \\ c_{n-1} & c_{n-2} & \dots & \dots & c_1 & c_0 \end{bmatrix}.$$

Fast algorithms for circulant matrices are based on the following spectral theorem: *Let C be any circulant matrix of order n with the first column c. Then*

$$C = \frac{1}{n} F^{-1} diag\ (Fc) F,$$

where F is the Discrete Fourier Transform matrix of order n. This theorem can be easily generalized to the block circulant (circulant-block) case. Due to the FFT, we obtain a direct solver for A with complexity $O(mn \log n)$ if A_i, B_i, and C_i have structure of G.

In general, A_i, B_i, and C_i have no structure. Instead of the above-discussed construction related with some (virtual) concentric circles, we propose to use the *optimal circulant-block preconditioner*. This preconditioner has the same form as the one with the concentric circles but can be defined entirely in the matrix language as the minimizer of the Frobenius-norm distance between A and all matrices of the above form. The preconditioner is now built up from the entries of A.

We found that the number of iterations with the optimal circulant-block preconditioner can be reduced by a factor of 10. Such a solver definitely outperforms the direct solvers based on the LU decomposition even for rather small n.

The idea of using circulants as preconditioners for Toeplitz matrices has been (and go on to be) addressed in many papers; we mention only [2] with a nice proof of superlinearity of Strang's circulants and [4] giving birth to the optimal circulants. The use of circulants and block circulants as preconditioners for general (non-Toeplitz) matrices has been studied in

[6, 7]; application of circulants for some matrices of the boundary integral method is discussed in [3]. Notice a detailed analysis of spectral equivalence and superlinearity in this case that can be found in [8].

5. Numerical results

Consider the case of three subdomains with relative resistivities 2, 10, 50 and boundaries depicted on Fig. 1. The disc inside the boot-shaped domain is of radius 0.1, and the boot-shaped boundary is given by a spline of degree 5 passing across the marked 10 points with the coordinates

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x = 1.0,  0.8,  0.5,  0.3,  0.2,  -0.3,  -0.4,  -0.3,  0.2,  0.7;
y = 0.0,  0.3,  0.4,  0.7,  0.9,   0.9,   0.5,  -0.2, -0.3, -0.3.
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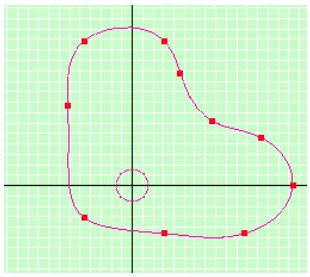

Figure 1. Boot-shaped domain with a disc inside.

The source is the magnetic z-dipole at the origin and we trace the z-component of magnetic field at some points on the z-axis near the origin. The operating frequency is set to 20 KHz.

We successively take $n_\Gamma = 32, 64, 128$ nodes on each boundary. In these cases the computation of the Fourier integrals was based upon $n_\zeta = 47, 39, 39$ adaptively chosen values of ζ, respectively. For each ζ we have to solve a linear system of size $n = 8n_\Gamma$.

Below we compare the results for unpreconditioned GMRES (with no limits on the subspace dimension) and GMRES with our circulant-block preconditioner; reported are the average numbers of iterations for all involved linear systems for different ζ. The residual reduction parameter used to quit iterations was set to 10^{-7}.

Table 1. Everage number of GMRES iterations

Number of nodes on each boundary	No preconditioner	Circulant-block preconditioner
32	$6409/47 \approx 136$	$425/47 \approx 9$
64	$5787/39 \approx 148$	$371/39 \approx 10$
128	$5387/39 \approx 138$	$371/39 \approx 10$

It is interesting that we do not observe the growth of the number of iterations as the matrix size increases. Nevertheless, the preconditioner reduces the number of iterations dramatically. It is especially important in this application because there are several tens of linear systems to be solved (each system corresponds to one value of ζ); the number of these systems may grow depending on the parameters of a particular application.

Not the least, one of the essentials of efficiency is the fast approximation method. Note that the piecewise constant approximations are noticably worse. Consider 2 disks with radii 0.3 and 0.5 (m). Let the resistivities of the corresponding domains be 10.0, 2.0, 1.0 and operating frequency be 10 (KHz).

Table 2. Piecewise constant approximations

Number of nodes	TIME (sec)	Relative error in H_z
16	3.3	13 %
32	10.0	8 %
64	39.3	5 %

In the same cases, our method takes 0.8 sec. with accuracy 0.7 % (with only 16 nodes on each contour).

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