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An Exact Analytical-Expression for the Read Sensor Signal in Magnetic Data Storage Channels

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1. Introduction

The new result described in this paper is an exact analytical expression for the Sensitivity Function of the Magnetic Read-Sensors currently used in Magnetic Data Storage Products, which are the digital memory devices used by most of the existing Information Technology Industry. The knowledge of this expression for the Non-Linear Magnetic Data Storage System is equivalent to the knowledge of the Impulse Response in a Linear System. From this familiar engineering viewpoint, the Sensitivity Function Sample Values are the sample values of the "Impulse Response" of the Magnetic Data Storage System, which is the Read Sensor response to a Recorded Magnetic Transition, storing one information bit.

The Readback Voltage Signal in, current, Perpendicular Magnetic Recording Hard Disk Drives is effectively determined by the Magneto Resistive (MR)-Sensor with Soft Underlayer (SUL) Geometry, as described in Figure 1

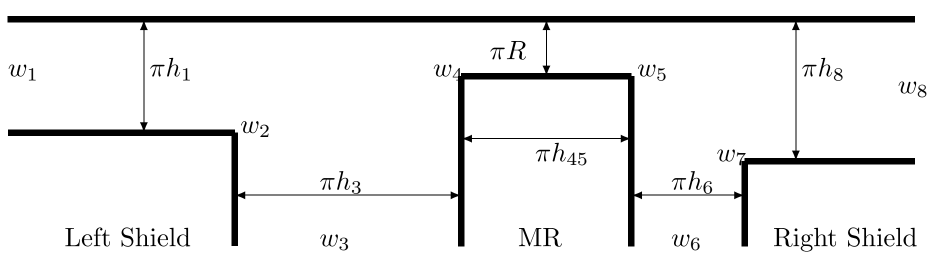

FIGURE 1. The dimensions of the octagonal gomain

The MR-Sensor Voltage Signal Output $V(t)$ is, approximately, its *Sensitivity Function* which is the imaginary component of the Magnetostatic Potential $H(w)$ which solves the Dirichlet Boundary Value Problem of the MR-SUL-Read Sensor Geometry.For MR-Sensors $V(t)$ expressions are given in [**4, 5, 6**]. The Sensitivity Function approximation can be made quite exact, once the nonlinear map inherent in the MR-Sensor, which converts magnetic flux variations into voltage variations, is experimentally measured and the signal is subjected to its inverse. If this is done then, for all practical purposes, the readback voltage function is $\psi(w) = Im(H(w))$, whose analytic computation is the subject matter of this paper. The possible arithmetical signal processing application of having voltage sample values as analytic function values is briefly outlined in the last section of this paper.

The calculation of the function $\psi(w)$ requires the solution of a Dirichlet Boundary Value Problem (BVP) which is described in the next section for the Perpendicular Recording System Read-Sensor. The method used is that of Schwartz-Christoffel Integral, $Eq(1)$ below, Conformal-Map, by which the 2-Dimensional Read Magnetic Sensor Geometry is mapped to the Upper Half Complex Plane. By a "lucky coincidence", this integral is a Doubly-Periodic Elliptic Integral, as initially introduced by Abel and Jacobi in order to rectify elliptical celestial trajectories. By the inversion of the Abel-Jacobi Elliptic Integral Period-Map onto the Period Rectangle, such an integral is "uniformized" in terms of Jacobi Theta-Functions, which are holomorphic power series, whose "zeros" are at the corners of the Elliptic Integral Period Rectangle, the "Jacobian", shown in $Figure.2$, of the Elliptic Curve, whose algebraic expression is provided by $Eq(2)$. In $Eq(3)$ appear Complete Elliptic Integrals of $3-rd$-Kind expressed in terms of Jacobi Theta Functions, these integrals were already used by Maxwell in his book "A Treatise on Electricity and Magnetism", vol.2, Ch.14, to provide analytical expression for the Magnetic Potential of a Current Loop. These are the "Magnetic Potential Units" in terms of which the Magnetic Storage Sensitivity Function is expressed, in $Eq.9$. The Magnetic Read-Sensor "Octagon Geometry", whose Dirichlet Boundary Value Problem is being solved in this paper, is portrayed in $Figure.1$. The MR-Sensor in this figure is a "Magnetic Dipole", the physical reason for the appearance of Elliptic Integrals of 3-rd Kind which are elementary potentials describing dipole configurations. This Magnetic Sensor Geometry is currently used in Perpendicular Magnetic Storage Products, where due to enhanced thermal stability properties, the Perpendicular has replaced Horizontal Magnetic Recording. The geometry includes a Soft Underlayer in the Disk as described in [**7**].

2. Analytical solution for the Magnetostatic Potential of the MR-Sensor with Soft Underlayer

2.1. Problem statement. Let us consider an Octagonal Domain, (in what follows, 'octagon' for short) , which is determined by six parameters: the width of the MR head πh_{45}; the distance from the left (resp. right) shield to the underlayer πh_1 (resp. πh_8); the gap between MR head and the left (resp. right) shield πh_3 (resp. πh_6) and distance from the underlayer to the MR element πR, obviously related to the recession parameter. The dimensions of Octagon as well as its corners are shown in Fig. 1. In practical applications we usually have $h_1 = h_8$.

The sensitivity function $\psi(w)$ is the function which is harmonic inside the octagon and takes the boundary value equal to 1 on the segments $[w_3, w_4]$, $[w_4, w_5]$, $[w_5, w_6]$ and 0 on the remaining part of the octagon boundary.

2.2. Mapping rectangle to the Octagon. First of all we give an explicit parametric representation for the conformal mapping $x(w)$ of the octagon to the half-plane. The inverse mapping is given by the Schwartz-Christoffel integral:

$$w(x) = Const \int^x \frac{\sqrt{(t-x_2)(t-x_4)(t-x_5)(t-x_7)}}{(t-x_1)(t-x_3)(t-x_6)(t-x_8)}\, dt, \tag{1}$$

with points x_s being the images of the corners w_s, unknown at the moment. Three of the unknown x_s may be removed by the normalization of the conformal mapping $x(w)$, while six remaining unknowns (including the constant) are related to six dimensions of the octagon.

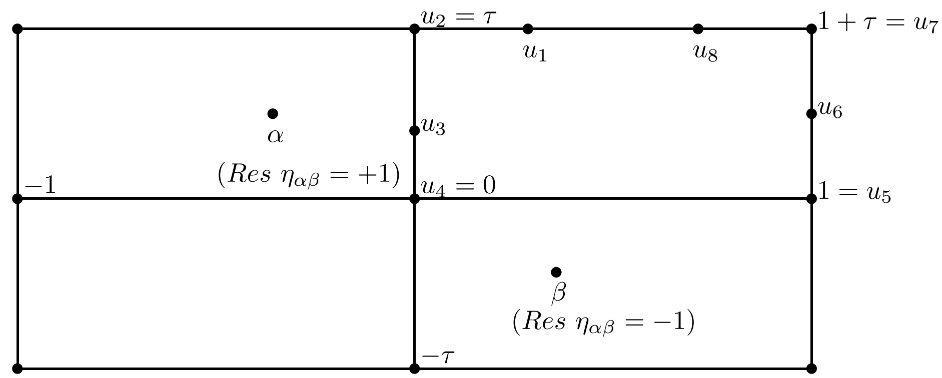

FIGURE 2. The torus (2) as a rectangle with identified sides

The differential $dw(x)$ is the third kind abelian differential on the torus

$$y^2 = (t - x_2)(t - x_4)(t - x_5)(t - x_7) \tag{2}$$

where it has eight simple poles (with projections x_1, x_3, x_6, x_8 to the x-plane) and four double zeroes located in the branchpoints of the curve. Now we consider another model of the torus (2), namely the factor of the comlex u-plane by the lattice $2\mathbb{Z} + 2\tau\mathbb{Z}$ with purely imaginary elliptic modulus τ. Elementary abelian integral of the third kind $\eta_{[\alpha,\beta]}(u)$ with simple poles at $u = \alpha$ (residue $= +1$) and $u = \beta$ (residue $= -1$) has a simple expression

$$\eta_{[\alpha,\beta]}(u) = \log \frac{\theta(\frac{u-\alpha}{2})}{\theta(\frac{u-\beta}{2})}, \qquad \theta(u) := \theta_{11}(u, \tau) = -2\exp(i\pi\tau/4)\sin(\pi u) + \dots \tag{3}$$

in terms of (the only) odd theta function of the modulus τ (see the definition in [**1, 2**]). Subtracting the terms like (3) with suitable singularities from the abelian integral $w(x(u))$, we obtain the holomorphic abelian integral on the torus. In other words,

$$\begin{aligned} w(u) = \quad & h_1 \log \frac{\theta((u-u_1)/2)}{\theta((u+u_1)/2)} \\ & -h_8 \log \frac{\theta((u-u_8)/2)}{\theta((u+u_8)/2)} \\ & +ih_3 \log \frac{\theta((u-u_3)/2)}{\theta((u+u_3)/2)} \\ & +ih_6 \log \frac{\theta((u-u_6)/2)}{\theta((u+u_6)/2)} \\ & +Cu, \end{aligned} \tag{4}$$

where the points u_1, u_3, u_6, u_8 are the positions of the poles of the differential dw, definition is clear from the Fig. 2. Six values C, $Re\ u_1$, $Re\ u_8$, $Im\ u_3$, $Im\ u_6$ and

$Im\ \tau$ in representation (4) are unknown. They satisfy the system of six equations:

$$dw(u)/du = 0, \qquad u = 0,\ 1,\ \tau,\ 1+\tau; \tag{5}$$

$$C = \pi(h_3 + h_{45} + h_6 + i(h_8 - h_1)) \tag{6}$$

$$R = Im\ \tau(h_3 + h_{45} + h_6) + h_8 + h_1 Re\ u_1 - h_8 Re\ u_8 - h_3 Im\ u_3 - h_6 Im\ u_6 \tag{7}$$

The first four equations (5) mean that dw has zero in every branchpoint of the curve, this zero will be double automatically. The fifth equation stems from integrating the differential dw from $u_2 = \tau$ to $u_7 = \tau + 1$. And (7) comes from evaluating the integral of dw from $u_4 = 0$ to $u_2 = \tau$. This system of equations has a unique solution (we do not prove it here) satisfying natural restrictions on the unknowns: $0 < Re\ u_1 < Re\ u_8 < 1$, $0 < Im\ u_3 < Im\ \tau$, $0 < Im\ u_5 < Im\ \tau$.

Two of the equations (5), (6), (7) for the auxiliary parameters of the mapping (4) are linear. Therefore, compared to the classical approach [**3**], we essentially have less number of equations for those auxiliary parameters. Moreover, all functions in formulas (4), (5) are effectively evaluated as theta function is represented by an extremely rapidly convergent series.

2.3. Numerical example. We consider the octagon with parameters $\pi h_1 = \pi h_8 = 49nm$, $\pi h_1 = 17.3nm$, $\pi h_{45} = 3.4nm$, $\pi h_6 = 14.3nm$, $\pi R = 45nm$. The u-images in the rectangle of the sets of w-points located in the octagon at the same altitude d and with equal horizontal spaces T are shown in the Fig 3.

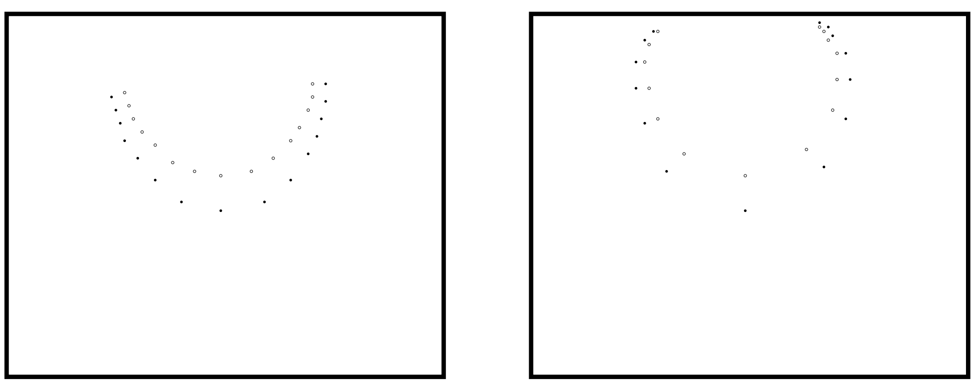

FIGURE 3. Images in the rectangle of four groups of points at the flying heights $d = 8nm$ and $d = 10nm$. Left picture: $T = 2nm$; right picture: $T = 5nm$

2.4. Mapping rectangle to the half-plane. The conformal mapping from the fundamental rectangle $\{0 < Re\ u < 1; 0 < Im\ u < |\tau|\}$ to the upper half-plane with normalization $x(u_3) = \infty$, $x(u_6) = 0$ is given by the standard formula [**1**]:

$$x(u) = \exp(\eta_{[u_6,u_3]}(u) + \eta_{[-u_6,-u_3]}(u)) = \frac{\theta((u-u_6)/2)\theta((u+u_6)/2)}{\theta((u-u_3)/2)\theta((u+u_3)/2)} \tag{8}$$

2.5. Magnetic potential. The boundary value problems for harmonic functions in the half plane may be solved analytically. In particular, our sensitivity function transferred to the rectangle is given by the explicit formula

$$\Psi(u) := \psi(w(u)) = \frac{1}{\pi} Arg\ x(u) = \frac{1}{\pi} Im\ (\eta_{[u_6,u_3]}(u) + \eta_{[-u_6,-u_3]}(u)), \tag{9}$$

where $x(u)$ is given above.

3. A Signal Processing Application

The Magnetic Readback Signal $V(t)$ is sampled at regular time intervals $t_k = kT$, where T is a "clock" period and the samples are processed to determine the recorded bits. In current magnetic signal processing methods the "shape" of a single magnetic transition readback signal is "equalized" to fit a "Partial-Response Signal". The "Signal Shapes" used are such that the samples $V(t_k)$ take prescribed integer values.The underlying idea is that the signal values, corresponding to a sequence of magnetic transitions, are predictable, as they are obtained by linear superposition of a known set of values, and can thus be described by a finite "trellis" graph, on which Maximum-Likelihood Decisions, using "Viterbi Algorithm", are performed to decode the stored information. This method, incorrrectly, replaces the actual nonlinear magnetic signal addition by linear superposition as an addition law for magnetic readback signals. Furthermore, the Viterbi Algorithm complexity increases exponentially as a function of the information channel memory, making this signal processing method impractical for the efficient processing of large data sector format, which is currently being adopted by the Hard Disk Drive (HDD) Industry.

The exact analytical expressions, in terms of Complete Elliptic Integrals of 3-rd Kind, for the Magnetic Sensitivity Function Values, shown in this paper, provide an "Analytical Signal Shape" for the magnetic transition readback signal whose sample values have a natural algebraic addition law, that can be efficiently formulated, as we want to point out.

Specifically,the signal values $\Psi(u_k)$ where u_k are the values in the Elliptic Jacobian corresponding to the sampling times t_k along the Magnetic Read Sensor "Flying Height" line, shown in Fig.3, can be approximated by rational lattice, division, points on the u-Plane Period Lattice in Fig.2, usually referred to as the Elliptic Curve Jacobian. The $\Psi(u_k)$-values will then be determined by the values of Elliptic Integrals of 3-rd kind, at rational division points in the Jacobian, described as follows

$$V(t_k) \approx \frac{1}{\pi} Im\ (\eta_{[u_6,u_3]}(u_k) + \eta_{[-u_6,-u_3]}(u_k)) \tag{10}$$

A sequence of recording magnetic transitions will produce a sequence of Readback Voltage Signals, which are combinations of sample values given by $Eq.10$. For u_k-Rational Division Points, these sample Values $V(t_k)$, given by $Eq.10$, are algebraic numbers in an algebraic number field.

The classical theorems of Abel and Jacobi provide Algebraic Addition Laws for Sums of Complete Elliptic Integrals, in particular those of 3-rd kind ["Ueber die Additionstheoreme der Abelschen Integrale zweiter und dritter Gattung", C.G.J.

Jacobi, Crelle's Journal, vol 30, p.121-126, 1845]. Given the values of such sums, they provide algebraic inversion formulas, by which,from given sum values, they calculate the Jacobian Coordinates of their compositions.

These Invertible Algebraic Addition Laws, can be used to construct "Algebraic-Addition" Trellis- Decoders, whereby sums of observed Readback Voltage Sample Values are mappped onto locations of magnetic transitions in the u-Plane Jacobian Lattice. These "Algebraic-Addition" laws, using algebraic number field arithmetic, which apply naturally to the Sensitivity Function Sample-Value provided in this paper, would thus replace the artificially imposed, and incorrect, Linear Superposition currently used in Magnetic Signal Processing.

4. Conclusion

We provide an explicit analytic expression for the Sensitivity Function Values of the Magnetic Read Sensors used in current Perpendicular Magnetic Recording HDD. We furthermore outline its possible application to the efficient decoding of stored information on HDD as a natural application of the classical Abel-Jacobi Algebraic Addition Laws it satisfies.

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