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Prime Form and Schottky Model

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Abstract. A new efficient variational formula for the Kleinian prime form (factor) in the frame of the Schottky model of Riemann surfaces is presented. We also give an elementary explanation for the choice of the sign in the transformation formula for the prime factor.

Keywords. Riemann surface, prime form, Schottky uniformization, variational formula.

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1. Introduction

The prime form was invented by F. Schottky [11] and F. Klein [10]. For hyperbolic surfaces it solves the problem of reconstruction of a meromorphic function from the set of its zeroes and poles. Linear functions and elliptic sigma (also theta) functions play the same role for the sphere and correspondingly for the torus. The prime form became very popular in the last decade among pure mathematicians, numerical community and theoretical physicists.

Let us recall the *classical definition* of the prime form. We fix two coordinate neighborhoods on a compact Riemann surface M of genus $g > 1$ and identify any points x, y in the neighborhoods with their coordinates. The following function is known as the Kleinian prime factor:

$$\Omega(x,y) := \left[\lim_{x'\to x,\, y'\to y} \frac{-(x-x')(y-y')}{\exp\left(\int_y^x d\eta_{x'y'}\right)} \right]^{1/2}, \tag{1}$$

here $d\eta_{xy}$ is the abelian differential on M with exactly two simple poles at the points x, y, and residues $+1$, -1 respectively. The differential is normalized

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so that its periods along all A-cycles of the fixed canonical basis (the so called *Schottky marking* of the surface) vanish. The normalization of the square root in (1) is determined by the asymptotics near the coinciding arguments:

$$\Omega(x,y) = (x-y)(1 + \text{higher order terms}). \tag{2}$$

The value $\Omega(x,y)$ of course depends on the choice of local coordinates near the arguments x, y. However, the inverse bispinor $E(x,y) := \Omega(x,y)/\sqrt{dx \cdot dy}$ is locally well defined and it is called *the prime form.* The prime form acquires simple non-vanishing multipliers when its arguments walk around the handles of the surface (see [5, 1, 6] and the last section of this note). So the natural domain of definition of the prime form is the squared universal covering of M.

2. Schottky model

The prime form has a well-known expression in terms of Riemann theta functions [1, 6]. However the first representation of the prime form was given in terms of the Schottky model of Riemann surfaces [11].

We consider $2g$ smooth simple loops A_j, A'_j, $j = 1, \dots, g$, on the plane, each lying in the exterior of all the others. Suppose that a linear fractional map $S_j(u)$ maps the interior of the loop A'_j to the exterior of the loop A_j. In this case S_j, $j = 1, \dots, g$, freely generate the Schottky group $\mathfrak{S}$. The fundamental domain $\mathcal{F}$ of the group is the exterior of all $2g$ contours. The generators S_j allow us to identify pairwise the boundary components of $\mathcal{F}$. The fundamental domain with those identifications becomes a compact Riemann surface M of genus g. Any Riemann surface may be represented as the orbit space of a suitable Schottky group $\mathfrak{S}$.

Let us give a representation of the prime form in terms of this model of the surface M. Fix two points x, y of the fundamental domain $\mathcal{F}$. An abelian differential $d\eta_{xy}$ of the third kind on M becomes a $\mathfrak{S}$-invariant differential form with simple poles in the orbits of x and y and normalized by

$$\int_{A_j} d\eta_{xy} = 0, \qquad j = 1, \dots, g.$$

When the limit set of the Schottky group has zero length (e.g. when all the generators $S_j(u)$ are real) the differential may be represented as an absolutely convergent linear Poincare series

$$\begin{aligned} d\eta_{xy}(u) &:= \sum_{S \in \mathfrak{S}} \left(\frac{1}{S(u) - x} - \frac{1}{S(u) - y} \right) dS(u) \\ &= \sum_{S \in \mathfrak{S}} \left(\frac{1}{u - S(x)} - \frac{1}{u - S(y)} \right) du. \end{aligned} \tag{3}$$

The passage of one representation of the differential $d\eta_{xy}$ to the other is due to the infinitesimal form of the cross-ratio identity.

Let us separate the poles of the differential $d\eta_{xy}$ which lie inside the fundamental domain

$$d\eta_{xy} = \frac{du}{u-x} - \frac{du}{u-y} + d\eta^*_{xy}.$$

The remnant $d\eta^*_{xy}$ of course is not $\mathfrak{S}$-invariant. But it is holomorphic in $\mathcal{F}$ and its definite integral depends on the endpoints only provided the path of integration remains within the fundamental domain. The latter property follows from the chosen normalization of $d\eta_{xy}$. We arrive at the following representation for the Kleinian factor:

$$\Omega(x,y) = (x-y)\exp\left(-\frac{1}{2}\int_y^x d\eta^*_{xy}\right), \qquad x, y \in \mathcal{F}. \tag{4}$$

Explicitly distinguishing the poles the differential $d\eta_{xy}$ of the third kind in a larger domain we obtain a similar representation valid in the fundamental domain $\mathcal{F}$ translated by the elements of the Schottky group. Finally, if the linear Poincare series for the Schottky group is absolutely convergent, the Kleinian factor may be represented in the entire domain of discontinuity as an infinite product

$$\Omega(x,y) = (x-y) \prod_{1\neq S\in\mathfrak{S}}{}' \frac{(x-S(y))(y-S(x))}{(x-S(x))(y-S(y))}.$$

The prime at the product symbol indicates that only one element in each pair of two inverse elements S, S^{-1} should be taken.

3. Variational formula

One of the advantages of the Schottky model of Riemann surfaces is the simple representation of moduli. The coefficients of the generators $S_j(u)$, $j = 1, \ldots, g$, may be taken as local moduli. The conjugation of all the generators by the same element of $PSL_2(\mathbb{C})$ surely keeps the complex structure of the surface intact. Thus we arrive at $3g-3$ local moduli [7]. Let $\mathfrak{S}$ be a Schottky group with generators S_1, S_2, ..., S_g represented in the matrix form

$$S_j(u) := \frac{a_j u + b_j}{c_j u + d_j} \longrightarrow \hat{S}_j := \begin{Vmatrix} a_j & b_j \\ c_j & d_j \end{Vmatrix} \in \mathrm{GL}_2(\mathbb{C}).$$

Any small perturbation of the matrix elements gives rise to a deformation of the group $\mathfrak{S}$ and to a variation of the value of the Kleinian prime factor $\Omega(u,z)$.

Theorem 1. *A deformation of a single generator $S_l(u)$ — with all other $S_j(u)$, $j = 1, \ldots, g$, $j \neq l$, being fixed — leads to the variational formula*

$$\delta\Omega(x,y) = \frac{1}{4\pi i}\Omega(x,y)\int_{A_l} (d\eta_{xy}(u))^2 \operatorname{tr}[\mathcal{M}(u)\cdot\delta\hat{S}_l\cdot\hat{S}_l^{-1}]/du + o. \tag{5}$$

Here the points x and y belong to $\mathcal{F}$,

$$\mathcal{M}(u) := \begin{Vmatrix} -u & u^2 \\ -1 & u \end{Vmatrix}$$

is the Hejhal matrix and $\mathcal{O} := \mathcal{O}(\|\delta S_l\|)$. The orientation of the contour A_l is counterclockwise.

Proof. We denote all the objects related to the unperturbed Schottky group by the superscript 0, e.g. $S_j^0(u)$, $\mathcal{F}^0$, A_j^0, $\Omega^0(x,y)$. Starting from representation (4) of the Kleinian prime factor, we get the following chain of transformations

$$\delta\Omega(x,y) := \Omega(x,y) - \Omega^0(x,y) = -\frac{1}{2}\Omega^0(x,y)\,\delta\left(\int_y^x d\eta_{xy}^*\right) + \mathcal{O}\left(\delta\int_y^x d\eta_{xy}^*\right). \tag{6}$$

Once the perturbation matrix $\delta\hat{S}_l$ is small, the fundamental domain $\mathcal{F}^0$ remains within the discontinuity domain of the group $\mathfrak{S}$ generated by S_j, $j \neq l$, and the perturbed generator corresponding to the matrix $\hat{S}_l + \delta\hat{S}_l$. We apply the Cauchy residue formula to the function

$$\delta\eta_{xy}^*(u) := \eta_{xy}^*(u) - \eta_{xy}^{*0}(u) = \int^u d\eta_{xy}^* - d\eta_{xy}^{*0} = \int^u d\eta_{xy} - d\eta_{xy}^0 =: \delta\eta_{xy}(u),$$

which is holomorphic in the unperturbed fundamental domain, and obtain

$$\delta\eta_{xy}^*(u) = (2\pi i)\delta\int_y^x d\eta_{xy}^* = \int_{\partial\mathcal{F}^0} \delta\eta_{xy}^*(u)\,d\eta_{xy}^0(u) = \int_{\partial\mathcal{F}^0} \delta\eta_{xy}(u)\,d\eta_{xy}^0(u) \tag{7}$$

$$= \sum_{j=1}^{g}\int_{\partial\mathcal{F}^0} \eta_{xy}\,d\eta_{xy}^0. \tag{8}$$

The passage from (7) to (8) is due to the equality $\eta_{xy}^0(S_j^0 u) - \eta_{xy}^0(u) = const$, and the normalization condition $\int_{A_j^0} d\eta_{xy}^0 = 0$. Hence

$$\begin{aligned}\delta\eta_{xy}^*(u) &= \sum_{j=1}^{g}\int_{A_j^0}\left(\eta_{xy}((S_j^0)^{-1}u) - \eta_{xy}(u)\right)\,d\eta_{xy}^0 \\ &= \sum_{j=1}^{g}\int_{A_j^0}\left(\eta_{xy}(S_j \circ (S_j^0)^{-1}u) - \eta_{xy}(u)\right)\,d\eta_{xy}^0.\end{aligned} \tag{9}$$

In the latter transformation we apply the same argument as above: the difference between $\eta_{xy}(S_j v)$ and $\eta_{xy}(v)$ is independent of v. Expanding the increment of the abelian integral of the third kind in the last formula with respect to the perturbation parameters, we find

$$\begin{aligned}\delta\eta_{xy}^*(u) &= -\int_{A_l^0} d\eta_{xy}(u)d\eta_{xy}^0(u)\,\mathrm{tr}[\mathcal{M}(u)\cdot\delta\hat{S}_l\cdot\hat{(}S_l^0)^{-1}]/du + \mathcal{O} \\ &= -\int_{A_l^0}(d\eta_{xy}^0(u))^2\,\mathrm{tr}[\mathcal{M}(u)\cdot\delta\hat{S}_l\cdot\hat{(}S_l^0)^{-1}]/du + \mathcal{O}.\end{aligned} \tag{10}$$

For the last transformation we used the uniform estimate

$$d\eta_{xy} - d\eta_{xy}^0 = \mathcal{O}(\|\delta \hat{S}_j\|)$$

on the contour A_l which may be obtained in the framework of quasiconformal mappings [2, 7]. This is the desired variational formula. ■

Remark. The variational formula (5) is computationally efficient. To see this, one has to decompose the quadratic differential $(d\eta_{xy})^2$ into meromorphic and holomorphic quadratic differentials represented as (relative) quadratic Poincare series. For the latter series, the integral in formula (5) is a finite expression due to D. A. Hejhal [8]. For implementation of this strategy in actual computations see [3, 4].

4. Transformation rule

Already F. Schottky ([11], see also [1, 5]) knew what happens to the prime factor when the group $\mathfrak{S}$ acts on its arguments:

$$\Omega(S_j x, y) = \Omega(x, y) \exp\left(\int_x^y d\zeta_j - \frac{1}{2}\kappa_{jj}\right) \sqrt{S_j'(x)}, \qquad j = 1, \ldots, g. \tag{11}$$

Here $d\zeta_j$ is the holomorphic differential on M with the normalization $\int_{A_s} d\zeta_j = 2\pi i \delta_{js}$ (integration is counterclockwise); $\kappa_{jj} := \int_w^{S_j w} d\zeta_j$ is its diagonal period. One can see that there are two sources of ambiguity in this formula. The first one is the choice of the integration path in the definition of the period, the other one is the choice of the square root. In other words, the integration path defines the sign of the square root or, equivalently, the lifting of the generator S_j to $SL_2(\mathbb{C})$. We are going to make this choice explicit.

4.1. Rotation of a smooth curve. Let C be a smooth oriented simple curve in the plane starting at the point x and ending at the point y. We associate two real numbers RotG, RotC to this curve and show that they coincide.

Definition. The *Gaussian rotation* $\mathrm{RotG}(C)$ is the increment of the tangent vector argument as we move from the starting point of the curve C to its endpoint. The *Cauchy rotation* $\mathrm{RotC}(C)$ is defined as

$$\mathrm{RotC}(C) = \mathrm{Im} \int_C \frac{du}{u-x} + \frac{du}{u-y} := \lim_{x' \to x, y' \to y} \mathrm{Im} \int_{x'}^{y'} \frac{du}{u-x} + \frac{du}{u-y}, \tag{12}$$

where the points x' and y' tend to their limits along the curve C.

Lemma 1. *If a smooth curve C is composed of two curves C_1 and C_2, then* $\mathrm{RotG}(C) = \mathrm{RotG}(C_1) + \mathrm{RotG}(C_2)$ *and* $\mathrm{RotC}(C) = \mathrm{RotC}(C_1) + \mathrm{RotC}(C_2)$.

Proof. Let z be the interior point of C, which is the endpoint of C_1 and the starting point of C_2. The statement is trivial for the Gaussian rotation. The Cauchy rotation of the curve C is the limit of the imaginary part of the following integral

$$\int_{x'}^{y'} \frac{du}{u-x} + \frac{du}{u-y} = \lim_{z_1\to z;\ z_2\to z} \left(\int_{x'}^{z_1} + \int_{z_2}^{y'} \right) \frac{du}{u-x} + \frac{du}{u-y}.$$

where $z_j \in C_j$, $j = 1, 2$, and the integration is along (portions of) the curve C. The latter two integrals can be written as

$$\left[\int_{x'}^{z_1} \left(\frac{du}{u-x} + \frac{du}{u-z} \right) + \int_{z_2}^{y'} \left(\frac{du}{u-y} + \frac{du}{u-z} \right) \right] \\ + \left\{ \int_{x'}^{z_1} \left(\frac{du}{u-y} - \frac{du}{u-z} \right) + \left(\int_{z_2}^{y'} \frac{du}{u-x} - \frac{du}{u-z} \right) \right\}.$$

After passing to the limits $z_1, z_2 \to z$ and $x' \to x$, $y' \to y$ the imaginary part of the expression in the square brackets gives $\mathrm{RotC}(C_1) + \mathrm{RotC}(C_2)$. It remains to show that the imaginary part of the integrals in curly brackets is zero in the limit.

Due to the Riemann's reciprocity law for the abelian integrals of the third kind on the sphere, the first of the integrals in the curly brackets is equal to

$$\int_{y}^{z} \frac{du}{u-x'} - \frac{du}{u-z_1}$$

once the integration paths of both integrals do not intersect. The non-singular part of the integrals disappears after passing to the limit while the singular part takes the form

$$\lim_{z_1\to z,\, z_2\to z} \mathrm{Im}\left(\int_{z}^{y} \frac{du}{u-z_1} - \int_{z_2}^{y} \frac{du}{u-z} \right) = \pi - \lim_{z_1\to z,\, z_2\to z} \mathrm{Arg}\, \frac{z_1 - z}{z_2 - z}$$

with $0 < \mathrm{Arg} < 2\pi$. The latter value is zero since our curve C is smooth. ■

Lemma 2. *Let the curve C and the segment $[x, y]$ connecting its endpoints bound a convex domain. Then* $\mathrm{RotG}(C) = \mathrm{RotC}(C)$.

Proof. Let $\angle x, \angle y \in (0, \pi)$ be the angles of the boundary of our convex domain at the points x and y respectively. The argument of the tangent vector to C is monotonic. Therefore, $\mathrm{RotG}(C) = \pm(\angle x + \angle y)$. Here the sign "$-$" is taken if the convex domain lies to the left of the oriented segment $[x, y]$, otherwise the sign is "$+$". The Cauchy rotation of the curve is equal to the same value $\pm(\angle x + \angle y)$. ■

Corollary (to Lemmata 1 and 2)**.** *Any smooth simple curve may be subdivided into linear segments and portions satisfying the conditions of Lemma 2.*

Therefore Gaussian and Cauchy rotations coincide for any curve C and we call them just the *rotation* $\operatorname{Rot}(C)$ of this curve.

4.2. Choice of the sign.

Definition. A smooth simple curve B_j, $j = 1, \dots, g$, contained in the fundamental domain $\mathcal{F}$ and connecting its boundary components is called a *barrier*, if and only if B_j and its translation $S_j B_j$ compose a whole smooth curve.

Let B_j be any barrier coming through the point x of the fundamental domain $\mathcal{F}$. Denote C the part of the smooth curve $B_j \cup S_j B_j$ starting at the point x and ending at $S_j(x)$. It is clear that $\operatorname{Rot}(C) \equiv \operatorname{Arg} S_j'(x) \mod 2\pi$.

Theorem 2. *Formula* (11) *holds for*

$$\kappa_{jj} := \int_C d\zeta_j \qquad \textit{and} \qquad \sqrt{S_j'(u)} := -|S_j'(u)|^{1/2} \exp\left(\frac{i}{2} \operatorname{Rot}(C)\right).$$

Proof. In what follows we fix the generator S_j, $j = 1, \dots, g$, and denote it by S for brevity; $d\zeta$ means the holomorphic differential $d\zeta_j$. The representation (4) of the Kleinian factor $\Omega(x, y)$ suggests the following analytic continuation to the domain $S\mathcal{F}$

$$\Omega(Sx, y) = (Sx - y) \exp\left(-\frac{1}{2} \int_y^{Sx} d\eta^*_{Sx\ y}\right), \qquad x, y \in \mathcal{F}, \tag{13}$$

where

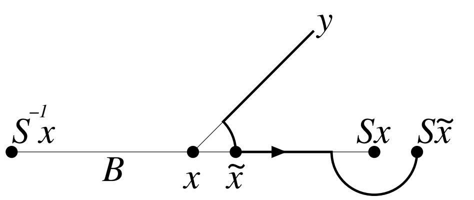
$$d\eta^*_{Sx\ y} = d\eta_{xy} + d\zeta - \frac{du}{u - Sx} + \frac{du}{u - y} = d\eta^*_{xy} + d\zeta - \frac{du}{u - Sx} + \frac{du}{u - x}.$$

The path of integration in (13) cannot be taken arbitrarily like in formula (4). Indeed, in the process of the analytic continuation we drag the endpoint x' of the integration path from x to Sx in such a way that the integration path does not meet the pole $S^{-1}x'$ of the differential $d\eta^*_{x'y}$. To secure ourselves from this possible nuisance we draw a *barrier* B passing through the point x. Now we fix a path from y to x in the fundamental domain crossing the barrier at the endpoint x only. The analytic continuation of formula (4) along the barrier B and its translation SB leads us to the following integration path in formula (13): from y to the vicinity of the pole x, make a detour around the pole to the point $\tilde{x} \in B$ and the path from $\tilde{x}$ to Sx along $B \cup SB$ (see Fig. 1 (a). Next we transform the integral in (13):

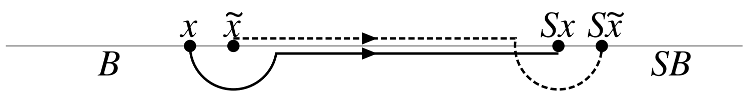
$$\int_y^{Sx} d\eta^*_{Sx\ y} = \lim_{\tilde{x} \to x} (I_1 + I_2), \tag{14}$$

where

$$I_1 = \int_y^{\tilde{x}} d\eta^*_{xy} + d\zeta - \frac{du}{u - Sx} + \frac{du}{u - x},$$
$$I_2 = \int_{\tilde{x}}^{S\tilde{x}} d\eta_{xy} + d\zeta - \frac{du}{u - Sx} + \frac{du}{u - y}.$$



(a) integration path for I_1 and I_2



(b) detour of poles $\tilde{x}$ and Sx.

Figure 1. Integration Paths.

Now we apply the reciprocity law for the differentials of the third kind on the sphere

$$\int_y^{\tilde{x}} \frac{du}{u - x} - \frac{du}{u - Sx} = \int_x^{Sx} \frac{du}{u - y} - \frac{du}{u - \tilde{x}}$$

(the integration paths should not intersect) and the Riemann identity

$$\int_{\tilde{x}}^{S\tilde{x}} d\eta_{xy} = \int_y^x d\zeta$$

(the fixed path from y to x and the detour around the pole x in the first integral should be at the opposite banks of the barrier as in Fig. 1(a). The limit in formula (14) now becomes

$$\int_y^x \left(d\eta^*_{xy} + 2d\zeta\right) + \int_C \left(d\zeta + 2\frac{du}{u - y}\right) - \lim_{\tilde{x} \to x} \left(\int_{\tilde{x}}^{S\tilde{x}} \frac{du}{u - Sx} + \int_x^{Sx} \frac{du}{u - \tilde{x}}\right).$$

The integrals under the limit sign have integration paths which make a detour around the poles at $\tilde{x}$ and Sx on the same side of C, opposite to the side where the fixed path from y to x approaches the barrier — see Fig. 1(b). The limit in the latter expression equals to

$$\log|S'(x)| \pm 2\pi i + i\,\mathrm{Rot}(C),$$

where the sign $\pm$ depends on the orientation of the intersection of the barrier with the fixed integration path from y to x. Combining the formulae we arrive at the transformation rule (11) for the prime factor with all the ambiguities being explained in the formulation of the theorem. ■

References

1. H. F. Baker, *Abel Theorem and the Allied Theory Including the Theory of Theta Functions*, Cambridge University Press, 1897.
2. L. Bers, Holomorphic differentials as functions of moduli, *Bull. Amer. Math. Soc.* **67** (1961), 206–210.
3. A. B. Bogatyrëv, *Extremal Polynomials and Riemann Surfaces*, Moscow, MCCME Publishers, 2005.
4. ———, Effective computation of Chebyshev polynomials for several intervals, *Math. Sbornik.* **190** no.11 (1999), 1571–1605.
5. W. Burnside, Further note on authomorphic functions, *Proc. London Math. Soc.* **23** (1892), 281–295.
6. J. Fay, *Theta Functions on Riemann Surfaces*, Springer, Lect. Notes in Math. **352**, 1973.
7. D. A. Hejhal, On Schottky and Teichmüller spaces, *Adv. Math.* **15** (1975), 133–156.
8. ———, Sur les parameters accessoires pour l'uniformization de Schottky, *C.R. Acad. Sc. Paris, Serie A* **279** (1974), 713–716.
9. M. Schiffer and N. S. Hawley, Half order differentials on Riemann surfaces, *Acta Math.* **115** (1966), 199–236.
10. F. Klein, Zur Theorie der Abel'schen Functionen, *Math. Ann.* **36** (1890), 1–83.
11. F. Schottky, Über eine specielle Function welche bei einer bestimmten linearen Transformation ihres Arguments unverändert bleibt, *J. Reine Angew. Math.* **101** (1887), 227–272.

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