Computations in moduli spaces

Andrei Bogatyrëv

Today the techniques related to Riemann surfaces are widely used in different branches of mathematics, theoretical physics, industry and even in medicine. Unfortunately, many achievements of the theory remain on paper since theoretical formulae contain special functions that few scholars try to compute. In the present note we describe several useful instruments for the solution of equations in the moduli spaces of algebraic curves. Conceptually all the methods we talk about belong to the 19th century but their actual implementation have become possible only with the invention of the computer .

1 Introduction

The solutions of numerous problems arising in mathematics, engineering, theoretical and mathematical physics are given in terms of Riemann surfaces. It is very often that one has to find a specific surface where certain relations among function -theoretic objects (e.g. meromorphic functions, linear or quadratic differentials, spinors) are satisfied. Essentially those relations give us the equations for *moduli* – the continuous parameters enumerating the conformal classes of complex curves. The reseachers who tried to resolve the constraints of this kind – see e.g. [6, 8, 13, 14, 21, 26] have developed many useful techniques. The first goal of this paper is to review those methods. For clarity we separate them from applications. The other goal is to popularize computations with Riemann surfaces. For the last reason we do not go into the details and give only ideas and sketches of the computational algorithms. Detailed description and proofs may be found from papers in the bibliography.

A noticeable part of publications on elliptic functions in the 19th century dealt with the computational aspects of the theory. Today the elliptic functions are widely used by non- mathematicians including engineers and the scholars of other disciplines. As to the higher genus curves, quite a narrow circle of researchers use them in computations or modelling. This is explained by two factors: a) the theory for hyperbolic curves is more involved than that for elliptic ones and b) computational methods and software for the genus $g > 1$ curves are still actively evolving. However, higher genus curves give us an adequate language to describe numerous phenomena so more and more scholars use them in their research.

Acknowledgements. This work was supported by RFBR grants 05-01-01027, 05-01-00582 and grant MD 2488.2005.1. The author is grateful to the anonymous referee who pointed at several recent works on the subject.

1.1 Three optimization problems

The applications of Riemann surfaces theory to the study of soliton equations like KdV, NLS, KP and the other are well known (see e.g. [5, 17, 22]). Somewhat less known are applications to optimization problems. We give three examples.

A. *Given a union E of several real segments, minimize $||P_n||_E := \max_{x \in E} |P_n(x)|$ among degree n monic polynomials $P_n(x) = x^n + \ldots$.*

This problem arises in optimization of numerical algorithms, signal processing and electrical engineering.

B. *Given points $a_1, a_2, \ldots, a_n$ in the complex plane, find a set $\mathcal{A}$ of minimal logarithmic capacity containing those points. (Additionally, one has to indicate which of the given points lie in the same component of the set $\mathcal{A}$.)*

Problem **B** is vital for the Padé approximations of algebraic functions.

C. *Among all polynomials of degree less or equal to n that approximate the exponential function at $x = 0$ to the third order, i.e. $P_n(x) = 1 + x + x^2/2 + x^3/6 + o(x^3)$, find the one that remains in the limits $-1 \le P_n(x) \le 1$ on the maximal interval* $[-L, 0]$, $L > 0$.

The last problem arises in construction of the multistage Runge-Kutta methods of 3rd order of accuracy. It was put forward approximately 50 years ago by several authors.

All the three problems may be reformulated as certain relations for the moduli of Riemann surfaces. Topologically, a compact Riemann surface is a sphere with several handles. The number g of handles has a special name – the genus. The conformal type of the genus g surface depends on $3g - 3$ complex parameters (moduli) when $g > 1$. The latter statement goes back to Riemann himself.

1.2 Reduction to moduli problem

The reduction of the problems **A** and **C** to the moduli problems may be found in [6] and [8] respectively.

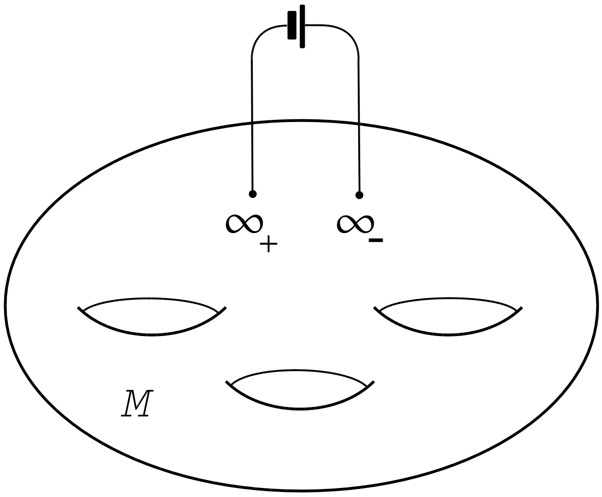

Figure 1: Electric potential on M gives rise to abelian differential $d\eta_M$

For the problem **B** we consider the hyperelliptic Riemann surface with fixed branch points a_s and several loose branch points b_j:

$$M = \{(x, w) \in \mathbb{C}^2 : \quad w^2 = \prod_{s=1}^{n} (x - a_s) \prod_{j=1}^{n-2l} (x - b_j)\}, \tag{1}$$

where $l = 1 + \sharp\{$ saddle points of Green's function for the set $\mathcal{A}$ $\}$. The genus of the curve M is $g = n - l - 1$.

On the Riemann surface M there exist the unique differential $d\eta_M$ with simple poles at both points above infinity, residues ± 1 and purely imaginary periods. In algebraic model (1) of the surface it has the appearance

$$d\eta_M = (x^g + c_1 x^{g-1} + \ldots + c_g)\frac{dx}{w}.$$

This distinguished differential has clear physical meaning used already by Riemann. Suppose that the surface M is made of conducting foil. We solder two infinitely thin wires to the two points $\infty_\pm$ of the Riemann surface that cover the point at infinity and connect them to the electric battery (see Fig. 1). The potential of the (stationary) electric current on the surface will be a harmonic function with two logarithmic poles. The conjugate harmonic function will be multi-valued on the surface M because of the poles and handles of M. Together, the potential and its conjugate function make up a multi-valued analytic function on the punctured surface – the abelian integral η_M. The differential $d\eta_M$ of the latter function is however single-valued and coincides with the distinguished differential on the surface M.

Nuttall conditions [25] for minimal capacity sets give $n - 2l$ equations for the same number of loose branch points b_j: they should be the zeroes of the distinguished differential $d\eta_M$ on the surface M. The minimal capacity set $\mathcal{A}$ now appears as the zero level of the real part of abelian integral $\int_{a_s}^x d\eta_M$ (= Green's function).

2 Moduli spaces $\mathcal{H}_g^k$ and $\mathcal{G}_g^k$

The moduli spaces of Riemann surfaces (with possibly marked points and boundaries) may have rather involved geometry and topology [24, 19]. For the general case we do not have a clear understanding of those spaces. In what follows we consider only the moduli of rather specific curves, namely real genus g hyperelliptic curves with $0 < k \le g+1$ real ovals and a marked point on the oriented real oval. Even this class of curves allows us to effectively solve a variety of interesting problems.

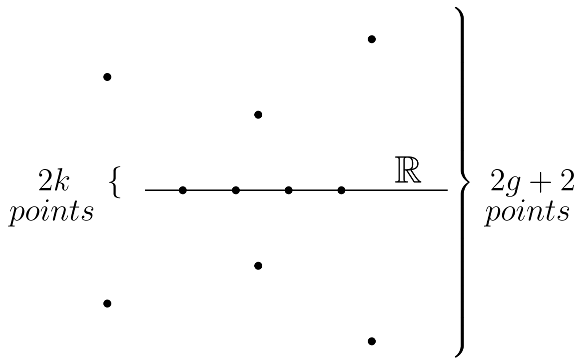

Figure 2: Symmetric branching divisor

We consider the unordered sets $\mathbf{e} = \{e_s\}_{s=1}^{2g+2}$ consisting of pairwise different $2k$ real points and $g-k+1$ pairs of complex conjugate points as in Fig.2 modulo action of 1D real affine group: $\mathbf{e} \to A\mathbf{e}+B$, $A > 0$. This set $\mathbf{e}$ serves as the branching divisor of two-sheeted (=hyperelliptic) Riemann surface

$$M(\mathbf{e}) := \{(x,w) \in \mathbb{C}^2 : \quad w^2 = \prod_{s=1}^{2g+2}(x-e_s)\}. \tag{2}$$

All the admissible sets $\mathbf{e}$ modulo the equivalence mentioned make up the moduli space $\mathcal{H}_g^k$. This space may also be thought of as the space of real genus g hyperelliptic curves with k real ovals (contours on the surface M where $(x,w) \in \mathbb{R}^2$, see Fig.3) and a marked point ∞_+ on the oriented real oval.

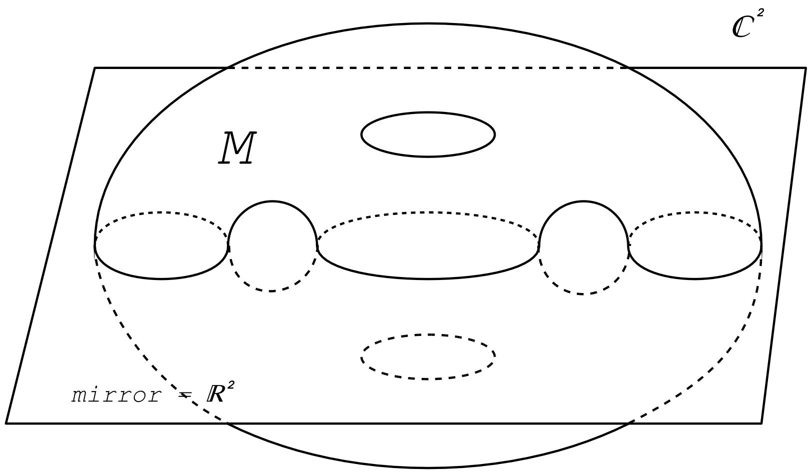

Figure 3: Real ovals of the curve M

Due to the mirror symmetry $\mathbf{e} = \bar{\mathbf{e}}$ the branching divisor is completely defined by its $2k$ real points as well as an unordered set of distinct $g - k + 1$ points in the open upper half plane $\mathbb{H}$. The two smallest real points may be sent to -1 and 0 by the unique normalizing linear map $x \to Ax + B$ with $A > 0$, so we get the representation

$$\mathcal{H}_g^k = \triangle^{2k-2} \times \left[\left(\mathbb{H}^{g-k+1} - \ \textit{diagonals}\ \right) / \{\textit{permutations}\} \right]$$

where $\triangle^{2k-2}$ is a simplex of dimension $2k - 2$ corresponding to varying real points in normalized branching divisor; *diagonals* stand for the points of $\mathbb{H}^{g-k+1}$ with two or more coinciding coordinates; *permutations* means the action of the permutation group on the coordinates of the point. The real dimension of the moduli space is $2g$ and its fundamental group $\pi_1(\mathcal{H}_g^k)$ is the Artin braid group on $g - k + 1$ strings – see Fig. 4.

Since the topology of the moduli space is not trivial, for many problems (say, solving equations on $\mathcal{H}_g^k$) it is useful to pass to the *universal covering space.* The latter is the set of paths considered up to homotopy and starting at the marked point of the moduli space. The universal covering space has an informal meaning as the "totally unrolled" space, for instance the totally unrolled torus is the plane.

Theorem 1 [7]: *The universal covering of the moduli space $\mathcal{H}_g^k$ is diffeomorphic to $\mathbb{R}^{2g}$.*

This theorem has the generalization: the universal covering of the moduli space is the Teichmüller space and the latter is always a cell.

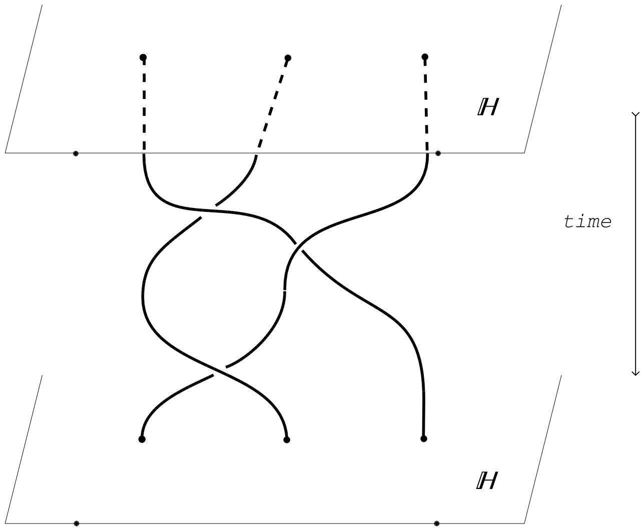

Figure 4: A loop in the moduli space $\mathcal{H}_3^1$ is a braid on 3 strings

2.1 The model of universal covering space

Let us consider g circles C_1, C_2, $\ldots$, C_g on the complex plane with non-overlaping real diameters lying on the interval $(0,1)$ – see Fig. 5. On every circle C_j we mark two points: real points $c_j \pm r_j$ when $j < k$ and complex conjugate points $c_j \pm ir_j$ when $j \geq k$. All ordered sets of real numbers $\{c_s, r_s\}_{s=1}^g$ obtained in this manner make up the *deformation space* $\mathcal{G}_g^k$. This space is a domain in the euclidean space of dimension $2g$ with coordinates c_s and r_s, $s = 1, \ldots, g$, satisfying certain inequalities [7].

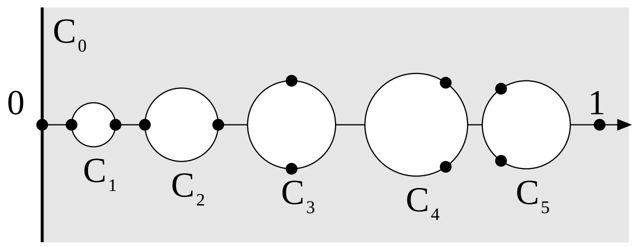

Figure 5: Fundamental domain of the group $\mathfrak{G}$ generated by a point of $\mathcal{G}_5^3$

Given a point in the deformation space we define the *Kleinian group* $\mathfrak{G}$ generated by $g+1$ linear-fractional transformations of order two:

$$G_s(u) := \begin{cases} -u, & s = 0, \\ c_s + r_s^2/(u - c_s), & s = 1, ..., k-1, \\ c_s - r_s^2/(u - c_s), & s = k, ..., g. \end{cases}$$

One can easily check that fixed points of the generator G_s are $c_s \pm r_s$ when $s = 1, 2, \ldots, k-1$ or $c_s \pm ir_s$ when $s = k, \ldots, g$ — the points we have marked on the circles. The group generators depend parametrically on the point in the deformation space while the group structure remains the same: $\mathfrak{G}$ is a free product of $g+1$ rank two groups.

The fundamental domain $\mathcal{R}$ of the group $\mathfrak{G}$ is bounded by the imaginary axis and circles C_s as in Fig. 5. Two marked points on C_s divide the circle into two arcs, one of them is mapped to the other by the generator G_s. In a similar way G_0 glues the positive imaginary semi-axis to the negative one. With all those identifications, the fundamental domain $\mathcal{R}$ becomes a sphere, so the orbit space of the group $\mathfrak{G}$ is the Riemann sphere.

Now we are going to describe the *projection* $\mathcal{G}_g^k \to \mathcal{H}_g^k$ and the action of *cover transformations* $\mathcal{G}_g^k \to \mathcal{G}_g^k$ which preserve this projection. Cover transformations make up the group isomorphic to the group of braids on $g-k+1$ strings.

2.2 Projection and cover transformations

The isomorphism of the orbit space of $\mathfrak{G}$ and the Riemann sphere is realized by an automorphic function $x(u): \quad \{\textit{domain of discontinuity}\} \to \mathbb{C}P^1$ with a single pole in fundamental domain $\mathcal{R}$. In what follows we normalize this map so that $x(u)$ sends points $u = 0,\ 1,\ \infty$ to respectively $x = 0,\ \infty,\ -1$. This map is unique and has mirror symmetry $x(\bar{u}) = \overline{x(u)}$. One can check that the branching divisor of the branched covering map $x(u)$, i.e. the set

$$\mathbf{e} = x(\textit{fixed points of} \quad G_0, G_1, \ldots, G_g), \tag{3}$$

consists of $2k$ real points (the images of $0, \infty$ and $c_s \pm r_s$, $s = 1, \ldots, k-1$) and $g-k+1$ pairs of complex conjugate points (the images of $c_s \pm ir_s$, $s = k, \ldots, g$). Therefore this divisor $\mathbf{e}$ is the (normalized) element of the moduli space $\mathcal{H}_g^k$.

Theorem 2 [7] *The mapping (3) from the deformation space $\mathcal{G}_g^k$ to the moduli space $\mathcal{H}_g^k$ is the universal covering map.*

Now we describe the action of cover transformations on the deformation space $\mathcal{G}_g^k$. Given a marked point $\mathbf{e}$ of the moduli space $\mathcal{H}_g^k$, the elementary braid β_s from the fundamental group $\pi_1(\mathcal{H}_g^k)$ interchanges two neighbouring branch points in the upper half plane as well as the symmetric branch points in the lower half plane. The corresponding cover transformation acts on the deformation space $\mathcal{G}_g^k$. It changes the set of generators of $\mathfrak{G}$:

$$(G_1, \ldots,\ G_{s-1},\ G_s,\ G_{s+1},\ G_{s+2}, \ldots,\ G_g) \xrightarrow{\beta_s}$$

$$(G_1, \ldots,\ G_{s-1}, G_s G_{s+1} G_s^{-1},\ G_s,\ G_{s+2}, \ldots,\ G_g), \qquad s = k, \ldots, g-1,$$

preserving the group itself and hence also the branched covering map $x(u)$ intact. This action is known as Artin representation of braids and may be easily rewritten in terms of the coordinates $\{c_s,\ r_s\}_{s=1}^g$ of the deformation space. It is a good exercise in elementary geometry to show that every braid maps the deformation space (i.e. a certain domain in $\mathbb{R}^{2g}$) to itself.

3 Constructive function theory

The nature of the moduli space $\mathcal{H}_g^k$ is twofold. On the one hand it is the space of divisors $\mathbf{e}$, on the other it is the space of real hyperelliptic curves M. Let us see how the Riemann surfaces appear in the model being discussed. Kleinian group $\mathfrak{G}$ contains index two subgroup $\mathfrak{S}$ generated by the hyperbolic linear fractional maps $S_j := G_j G_0, \quad j = 1, \ldots, g$. These generators are free (i.e. not subjected to any relations like $G_s^2 = 1$). The elements S of the Schottky group $\mathfrak{S}$ are finite words with the alphabet containing all the generators S_j and their inverse. We can introduce the norm $|S|$ of the element as the length of the irreducible word representing S, say $|S_1^{-1} S_2^2 S_1| = 4$.

Schottky group $\mathfrak{S}$ has the fundamental domain $\mathcal{R}_S := \mathcal{R} \cup\ G_0 \mathcal{R}$ and every generator S_j glues the boundary circle $G_0 C_j$ to the circle C_j. Hence the orbit space of the Schottky group is a genus g Riemann surface additionally endowed with the hyperelliptic involution $Ju := -u$ and the anti-conformal involution (reflection) $\bar{J}u := \bar{u}$. The marked point ∞_+ on the real oval is the point $u = 1$.

Our immediate goal is to represent various objects of the function theory in an effectively computable way when the curve M is given as the space of orbits of the Schottky group.

3.1 Abelian differentials in Schottky model

We start from abelian differential with two simple poles as the only singularities on M. The idea (H.Poincare, 1882) is to take a rational differential on the sphere with the same singularities and average it with respect to the linear fractional mappings from the Schottky group.

$$d\eta_{zz'}(u) := \sum_{S \in \mathfrak{S}} \left\{ \frac{1}{Su - z} - \frac{1}{Su - z'} \right\} dS(u) =$$

$$= \sum_{S\in\mathfrak{S}} \left\{ \frac{1}{u - Sz} - \frac{1}{u - Sz'} \right\} du. \qquad (4)$$

The passage of one representation of the differential $d\eta_{zz'}$ to the other is due to the infinitesimal form of the cross-ratio identity.

To obtain a holomorphic differential on M let us place poles z, z' in the previous formula into the same orbit of the group $\mathfrak{S}$ and consider the 'telescopic' sum:

$$d\zeta_j(u) := d\eta_{S_j z\ z} = \sum_{S\in\mathfrak{S}} \left\{ (u - SS_j z)^{-1} - (u - Sz)^{-1} \right\} du =$$

$$\sum_{S\in\mathfrak{S}|\langle S_j\rangle} \sum_{m=-\infty}^{\infty} \left\{ (u - SS_j^{m+1} z)^{-1} - (u - SS_j^m z)^{-1} \right\} du =$$

$$= \sum_{S\in\mathfrak{S}|\langle S_j\rangle} \left\{ (u - S\alpha_j)^{-1} - (u - S\beta_j)^{-1} \right\} du =$$

$$= \sum_{S\in\langle S_j\rangle|\mathfrak{S}} \left\{ (Su - \alpha_j)^{-1} - (Su - \beta_j)^{-1} \right\} dS(u), \quad j = 1, \ldots, g. \qquad (5)$$

Here α_j is the attractive and β_j is the repulsive fixed points of generator $S_j(u)$; the summation is taken over (arbitrary) representatives of the right or left cosets of the Schottky group $\mathfrak{S}$ with respect to the infinite cyclic subgroup $\langle S_j\rangle$ generated by S_j.

Differential of the 2nd kind (with poles of higher orders) may be obtained by taking derivatives with respect to the location of the simple pole:

$$d\omega_{mz}(u) := D_z^m d\eta_{zy}(u) = m! \sum_{S\in\mathfrak{S}} (Su - z)^{-m-1} dS(u), \qquad m = 1, 2, ...$$

Direct term by term integration of the series shows that the differentials thus obtained will automatically have the following normalization:

$$\int_{C_s} d\eta_{zz'} = 0; \quad \int_{C_s} d\omega_{mz} = 0; \quad \int_{C_s} d\zeta_j = 2\pi i \delta_{sj}; \qquad s, j = 1, \ldots, g. \qquad (6)$$

provided the poles z, z' are contained in the fundamental domain $\mathcal{R}_S$ specified above of the Schottky group bounded by the circles $\pm C_j$, $j = 1, \ldots, g$.

REMARKS:

1. Other Poincare theta series result in different function-theoretic objects like spinors (weight -1) and quadratic differentials (weight -4).

2. The convergence (rate) of Poincare theta series was investigated by H.Poincare, W.Burnside, R. Fricke, P.J.Myrberg, H.Petersson, A.F.Beardon, T.Akaza, D.Sullivan and many other authors. For the arbitrary Schottky

group the Poincare theta series we have used for representation of various differentials on the curve M is not absolutely convergent. Counterexamples may be found in [23, 1]. The absolute convergence of the *linear* Poincare theta series (= automorphic form of the weight -2) is equivalent to the zero length (1-Hausdorf measure) of the limit set of the group. The most simple criterion for the absolute convergence belongs to F.Schottky [27]: *the fundamental domain of the group may be decomposed into "pairs of pants" with circular boundaries.* Other criteria – see e.g. [26, 2], are formulated in terms of certain inequalities between the moduli of the group. The general idea of all criteria: *boundary circles of the fundamental domain should be well separated compared to their radii.* In the case under consideration the Schottky group has real generators and no question about convergence arises. However the problem with the convergence *rate* remains.

3.2 Schottky functions and Schottky-Klein prime factor

The following two functions with their arguments belonging to the discontinuity domain of the Schottky group were introduced by F.Schottky in 1887 [27]. Later W.Burnside in 1892 related those functions to the Poincare series [11].

$$(u, u'; z, z') := \exp\int_{u'}^{u} d\eta_{zz'} = \prod_{S\in\mathfrak{S}} \frac{u - Sz}{u - Sz'} : \frac{u' - Sz}{u' - Sz'}, \tag{7}$$

$$E_j(u) := \exp\int_{\infty}^{u} d\zeta_j = \prod_{S\in\mathfrak{S}|\langle S_j\rangle} \frac{u - S\alpha_j}{u - S\beta_j}, \quad j = 1, \dots, g.$$

Schottky functions are subjected to the obvious transformations under the permutations of their arguments and the substitutions from the group $\mathfrak{S}$:

$$(u, u'; z, z') = (z, z'; u, u') = 1/(u', u; z, z') = 1/(z', z; u, u')$$

$$(S_j u, u'; z, z') = (u, u'; z, z')E_j(z)/E_j(z'), \quad j = 1, \dots, g, \tag{8}$$

$$E_l(S_j u) = E_l(u)E_{lj}, \quad l, j = 1, \dots, g, \tag{9}$$

where the constant $E_{lj} := \exp(\int_u^{S_j u} d\zeta_l)$ admits the following representation

$$E_{lj} = E_{jl} = \prod_{S\in\langle S_l\rangle|\mathfrak{S}|\langle S_j\rangle} \frac{S\alpha_j - \alpha_l}{S\alpha_j - \beta_l} : \frac{S\beta_j - \alpha_l}{S\beta_j - \beta_l}, \quad l, j = 1, \dots, g. \tag{10}$$

Here the product is taken over the two-sided cosets of the group $\mathfrak{S}$ and the factor $0/\infty$ corresponding to $S = 1$ when $j = l$ is replaced by the stretch coefficient $\lambda_l := S_l'(\alpha_l)$.

In the same paper [27] F.Schottky introduced the function of a more involved nature, later used by F. Klein (1888, 1890), W.Burnside (1892), H.Baker (1897) and many others.

$$\Omega(u,z) := \left[\lim_{u'\to u,\ z'\to z} \frac{-(u-u')(z-z')}{\exp(\int_z^u d\eta_{u'z'})}\right]^{\frac{1}{2}} =$$

$$= (u-z) \prod_{1\neq S\in\mathfrak{S}}{}' \frac{(u-Sz)(z-Su)}{(u-Su)(z-Sz)} = -\Omega(z,u), \qquad (11)$$

where in the last product only one in the pair of two inverse elements S, S^{-1} should be taken. The function $\Omega(z,u)$ is known as Schottky-Klein prime factor (or function) and is closely related to the *prime form* [18]: $E(u,z) = \Omega(u,z)/\sqrt{du\cdot dz}$. Schottky function (7) is expressed in terms of the prime factor as

$$(u,u';z,z') = \frac{\Omega(u,z)\Omega(u',z')}{\Omega(u,z')\Omega(u',z)}.$$

The substitutions of the group $\mathfrak{S}$ transform the prime factor in the following way:

$$\Omega(S_j u,z) = \pm\Omega(u,z)[S'(u)/E_{jj}]^{\frac{1}{2}} E_j(z)/E_j(u), \qquad (12)$$

for the choice of sign in the last formula see [4].

The significance of the Schottky functions $(u,u';z,z')$, $E_j(u)$ and the prime factor $\Omega(u,z)$ for the function theory on the Riemann surface M is clear from the following result.

Lemma *Let $F(u)$ be any automorphic (with respect to the Schottky group $\mathfrak{S}$) function and $\sum_{s=1}^{\deg F}(z_s - z_s')$ be the divisor of $F(u)$ in the fundamental domain $\mathcal{R}_S$ of the group. The function may be recovered up to a constant factor by any of two formulae:*

$$F(u) = const^* \prod_{s=1}^{\deg F} (u,*;z_s,z_s') \prod_{j=1}^{g} E_j^{m_j}(u), \qquad (13)$$

$$F(u) = const \prod_{s=1}^{\deg F} \frac{\Omega(u,z_s)}{\Omega(u,z_s')} \prod_{j=1}^{g} E_j^{m_j}(u), \qquad (14)$$

where $m_s \in \mathbb{Z}$ – is the increment of $(2\pi i)^{-1}\log F(u)$ along the circle C_s and $*$ is an arbitrary point affecting only the value of the constant factor.

REMARK: The conditions of the automorphy of the right hand sides of the formulae (13), (14) are:

$$\prod_{s=1}^{\deg F} \frac{E_l(z_s)}{E_l(z'_s)} \prod_{j=1}^{g} E_{jl}^{m_j} = 1, \qquad l = 1, \dots, g.$$

They are equivalent to the Abel conditions for the divisor of the meromorphic function:

$$\sum_{s=1}^{\deg F} \int_{z'_s}^{z_s} d\zeta_l + \sum_{j=1}^{g} \Pi_{jl} m_j + 2\pi i m'_l = 0, \qquad l = 1, \dots, g,$$

with certain integer numbers m'_l, and where $\Pi_{jl} := \int_x^{S_l x} d\zeta_j$ is the period matrix of abelian differentials.

EXAMPLE. The above normalized projection of the discontinuity domain of the group onto the Riemann sphere takes the form

$$x(u) = -(u, \infty; 0, 1)(u, \infty; 0, -1).$$

Combining the last formula with (3) we obtain the effective evaluation of the projection from the deformation space $\mathcal{G}_g^k$ to the moduli space $\mathcal{H}_g^k$.

3.3 Computations with Schottky groups

The evaluation of Abelian differentials $d\eta_{zz'}$, $d\zeta_s$, $d\omega_{mz}$ or the Schottky functions $(u, u'; z, z')$, $E_j(u)$ requires summation or multiplication of terms enumerated by the elements of the Schottky group $\mathfrak{S}$. The group is infinite so we have to use some finite part of it for calculations. Here two problems arise: 1) to estimate (a priori or a posteriori) the value of the residual terms and 2) to minimize the amount of computational work and the memory used.

To organize the computation we built the Cayley graph of our group $\mathfrak{S}$. We put the group elements S into the nodes of the graph and connect them to the elements $S_j S$, $S_j^{-1} S$, $j = 1, \dots, g$. We get an infinite tree with valency $2g$ at every node. The tree is naturally decomposed into layers of nodes S with equal norm $|S|$ – see Fig. 6 (the nodes from the third layer are only partly shown). The same picture is used to explain what the nuclear chain reaction is.

EXAMPLE. For evaluation of the differential $d\eta_{zz'}$ we add the terms in the first series of (4). To calculate the terms corresponding to the nodes of the layer $|S| = n$ of the Cayley graph we have to keep only the values Sz, Sz' in the nodes of the previous layer $|S| = n - 1$. In particular there is no need

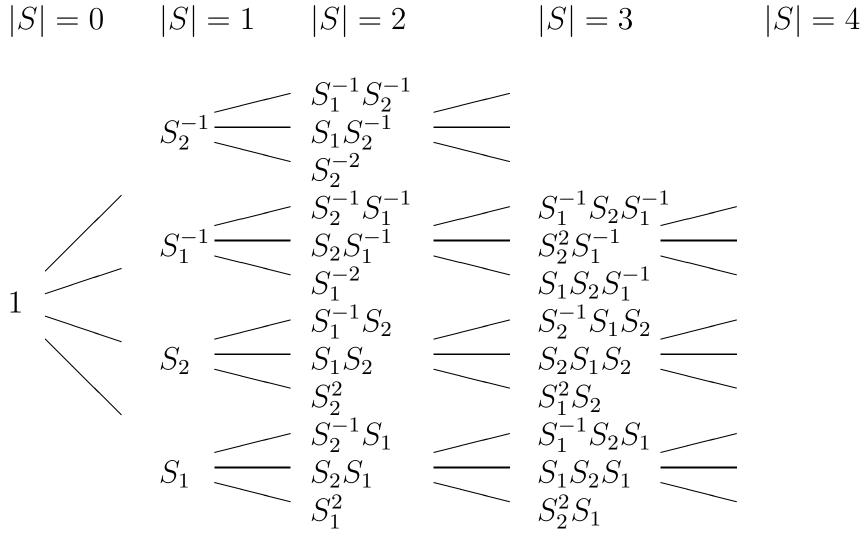

Figure 6: Cayley graph of the Schottky group for $g = 2$

to explicitly calculate the group elements S. Once we stop the summation on a certain layer of the tree, we can give an *a priori* estimate of the absolute values of all the dropped terms [6, 9]. However this algorithm is not very economic. The terms of the Poincare series corresponding to the same layer of the tree may be of different orders of magnitude due to the degeneracies of the group. It would be silly to spend the computational power for the summation of infinitesimal (less than machine accuracy) terms. To take this into account, we prune away the entire subtree growing from the node S of the Cayley graph when the absolute value of the corresponding to S term of Poincare series is lower than the given accuracy ϵ. This algorithm gives a dynamic choice of the finite subtree of the Cayley graph for computation and an *a posteriori* estimate of the error for it may be found in [9].

4 Variational formulae

The purpose of computations in the moduli space is to find Riemann surfaces with certain relations between functions or differentials holding on them. The examples of such computations may be found e.g. in [6, 8, 13, 14, 26]. We have learned from Section 3.2 that all meromorphic functions on the surface M are expressed in terms of Schottky (-Klein) functions and the latter are reduced to the case of abelian integrals. To solve equations in the moduli space by the Newton method it is vital to know how abelian integrals vary with the variations of coordinates $\{c_s, r_s\}_{s=1}^{2g}$ in the moduli space.

Let $\mathfrak{S}$ be the Schottky group with generators S_1, S_2, $\ldots$, S_g represented in the matrix form:

$$S_j(u) := c_j \mp r_j^2/(u+c_j) \longrightarrow \hat{S}_j := \begin{Vmatrix} c_j & c_j^2 \mp r_j^2 \\ 1 & c_j \end{Vmatrix}.$$

The deformation of the matrix elements gives rise to the deformation of the group $\mathfrak{S}$ and to variation of the values of abelian integrals and their periods.

Theorem 3 *The following variational formulae are valid for the definite abelian integrals [6, 9]:*

$$\delta \int_v^{v'} d\eta = (2\pi i)^{-1} \sum_{l=1}^{g} \int_{C_l} d\eta(u) d\eta_{vv'}(u) \mathrm{tr}[\mathcal{M}(u) \cdot \delta\hat{S}_l \cdot \hat{S}_l^{-1}]/du + o, \qquad (15)$$

their periods:

$$\delta \int_{S_j w}^{w} d\eta = (2\pi i)^{-1} \sum_{l=1}^{g} \int_{C_l} d\eta(u) d\zeta_j(u) \mathrm{tr}[\mathcal{M}(u) \cdot \delta\hat{S}_l \cdot \hat{S}_l^{-1}]/du + o, \qquad (16)$$

and for the prime factor [10]

$$\delta\Omega(x,y) = (4\pi i)^{-1} \Omega(x,y) \sum_{l=1}^{g} \int_{C_l} (d\eta_{xy}(u))^2 \mathrm{tr}[\mathcal{M}(u) \cdot \delta\hat{S}_l \cdot \hat{S}_l^{-1}]/du + o. \quad (17)$$

Here $d\eta(u)$ is any of the differentials $d\eta_{zz'}$, $d\zeta_s$, $d\omega_{mz}$; the endpoints $v, v' \in \mathcal{R}_S$; and $\mathcal{M}(u) := \begin{Vmatrix} -u & u^2 \\ -1 & u \end{Vmatrix}$ is the Hejhal matrix; $o := O(\sum_s (\delta c_s)^2 + (\delta r_s)^2)$.

REMARKS: 1) The proof of our variational formulae follows the deduction of the Hadamard formula for the variation of the Green's function. Similar formulae may be found in the works of L.Ahlfors, H.E.Rauch, M.Schiffer and D.C.Spencer.

2) Variational formulae hold for the arbitrary Schottky group even if the Poincare series from Section 3.1 do not converge. In this case instead of seria we take the differentials with the same singularities and normalization conditions (6).

The integrals at the right hand sides of variational formulae resemble Eichler periods for quadratic differentials $d\eta d\eta_{vv'}$, $d\eta d\zeta_j$. They may be effectively calculated in the following way. On the surface M there are quadratic differentials $\Xi(u)(du)^2$ with arbitrary singularities as well as the complete set of holomorphic ones represented by *quadratic* Poincare theta series over the Schottky group $\mathfrak{S}$. For the series the mapping

$$\Xi(u)(du)^2 \longrightarrow \int_{C_l} \Xi(u)\mathcal{M}(u)\, du,$$

from the right hand sides of variational formulae is explicitly given by the finite term Hejhal formulae [20, 6, 9]. What remains is to decompose the quadratic differentials like $d\eta d\eta_{vv'}$ into the basic quadratic differentials. The details may be found in [6, 9]. In a sense, the effective variational formulae give us a reliable method of navigating in the moduli space.

EXAMPLE. The following relative quadratic Poincare seria

$$\begin{array}{rl} \Theta_2^j(du)^2 := & \sum\limits_{S\in\langle S_j\rangle|\mathfrak{S}} R(Su)(dS(u))^2, \\ R(u) := & (u-\alpha_j)^{-2}(u-\beta_j)^{-2}; \end{array} \qquad j=1,\dots,g, \tag{18}$$

represent g of $3g-3$ holomorphic quadratic differentials on the surface M. All integrals from right hand sides of the variational formulae may be computed explicitly for those seria:

$$\int\limits_{\mathbf{C}_s} \Theta_2^j(u)\mathcal{M}(u)du = \frac{i\pi}{2\alpha_j^2}\begin{pmatrix} 0 & \alpha_j \\ \alpha_j^{-1} & 0 \end{pmatrix}\delta_{js}. \tag{19}$$

5 Conclusion

In this survey we have outlined the usage of Schottky uniformization for effective computations in the space of real hyperelliptic curves. The approach presented is readily extended to the spaces of general *real* algebraic curves. The mirror symmetry of the Riemann surfaces M (denoted here by the word "real") is however crucial. For general Schottky group Poincare series representing various linear differentials on the surface do not converge. The merit of this approach is in the explicit and easy-to-handle representation of the moduli of curves. The disadvantage is the low convergence rate for the Poincare theta series in the case of high genus and/or degeneracy of curves.

The alternative approach to effective computation uses Riemann theta functions. Here we have no limitations like mirror symmetry. The principal advantage of this approach consists in the high convergence rate of the Riemann theta series. The principal drawback is the representation of moduli. The latter appears as a certain manifold in the Siegel space of symmetric matrices with positive definite imaginary part. For a particular applied problem one has $3g-3$ equations on moduli and extra $g(g-1)/2-3g+3$ equations for the solution of the notorious Schottky problem. At the certain stage the complexity of the solution of those "additional" equations exceeds the computational cost of the solution of "essential" part describing the problem.

The benefits of both approaches are combined in the hybrid methods (e.g. [26]) when we compute the period matrix Π_{jl} in the Schottky model and then express all our function-theoretic objects in terms of the Riemann theta function.

References

[1] Akaza T. *Poincare theta series and singular sets of Schottky groups*//Nagoya Math. J. 24(1964), pp. 43–65.

[2] Akaza T. *Singular sets of some Kleinian groups*//Nagoya Math. J. 29(1967), pp. 145–162.

[3] Akaza T., Inoue K. *Limit sets of a geometrically finite free Kleinian group*//Tohoku Math. J 36 (1984), pp. 1–16.

[4] Baker H.F. *Abel theorem and the allied theory including the theory of theta functions* – Cambrigde University Press, 1897.

[5] Belokolos E.D., Bobenko A.I., Enol'skii V.Z., Its A.R., Matveev V.B. *Algebro-Geometric Approach to Nonlinear Integrable Equations* – Springer, Berlin, 1994.

[6] Bogatyrev A. *On evaluation of Chebyshev polynomials on several segments.* //Math. Sbornik, 190:11 (1999), pp. 15-50, LMS translations: 1571 – 1605.

[7] Bogatyrev A.B. Representations of the moduli spaces and computation of extremal polynomials //Sbornik: Math. 194:4 (2003), pp. 3-28, LMS translation: 469–494.

[8] Bogatyrev A.B. *Effective solution of a problem about optimal stability polynomials* //Sbornik: Math 196:7 (2005), pp. 27-50, LMS translations: 959-981.

[9] Bogatyrev A.B. *Extremal Polynomials and Riemann Surfaces* (in Russian) – Moscow, MCCME publishers, 2005.

[10] Bogatyrev A.B. *Effective variational formula for Schottky-Klein prime factor* – to appear

[11] Burnside W. *On a class of authomorphic functions* //Proc. London Math. Soc. vol XXIII, (1892), pp. 49-88.

[12] Burnside W. *Further note on authomorphic functions* //Proc. London Math. Soc. vol XXIII, (1892), pp. 281-295.

[13] Crowdy D. *Genus N algebraic reductions of the Benney hierarchy within a Schottky model* // J. Phys. A: Math. and General, 38 (2005), 10917-10934.

[14] Crowdy D., Marshall J.S. *On the construction of multiply connected quadrature domains* // SIAM J. Appl. Math., 64 (2004), 1334-1359.

[15] Deconinck B., Heil M., Bobenko A.I. et al. *Computing Riemann theta functions* – Math. Comp., 73 (247), 1417-1442, 2004.

[16] Deconinck B., Van Hoeij M. *Computing Riemann matrices of algebraic curves* – Physica D., 152, 28-46, 2001.

[17] Dubrovin B.A. *Theta functions and nonlinear equations* //Russ. Math. Surv. 36:2, 11-92, 1981.

[18] Fay J. *Theta functions on Riemann Surfaces* – Springer, Lect. Notes in Math.-352, 1973.

[19] Harris J., Morrison I. *Moduli of curves* – Springer-Verlag, 1998.

[20] Hejhal D.A. *Sur les parameters accessoires pour l'uniformization de Schottky*// C.R. Acad. Sc. Paris, t.279 (1974) Serie A pp.713–716.

[21] Hidalgo R.A., Figueroa J. *Numerical Schottky uniformizations* // Geometriae Dedicata, 111 (2005), 125-157.

[22] Krichever I.M., Novikov S.P. *Periodic and almost periodic potentials in the inverse problems* – ArXive: math-ph/003004, 2000.

[23] Myrberg P.J. *Zur Theorie der Convergenz der Poincareschen Reihen.*// Ann. Acad. Sci. Fenn. (A)9:4 (1916) S. 1–75; 11:4 (1917) S. 1-29.

[24] Natanzon S.M. *Moduli of Riemann Surfaces, Real Algebraic Curves, and Their Superanalogs* – AMS, 2004.

[25] Nuttall J. *Sets of minimum capacity, Padé approximants and the bubble problem* – in C.Bardos, D.Bessis (eds.) "Bifurcation phenomena in Math. Physics and related topics", pp. 185-201, D.Reidel Publ. Company, 1980.

[26] Schmies M. *Computational methods for riemann surfaces and helicoids with handles* // PhD thesis, TU Berlin , 2005.

[27] Schottky F. *Über eine specielle Function welche bei einer bestimmten linearen Transformation ihres Argumenten unverändert bleibt* // J. Reine Angew. Math. 101(1887), S. 227–272.

[28] Wang Y., Gu X., Hayashi K.M. el al. *Brain Surface Parametrization Using Riemann Surfaces* — Springer Verlag, Berlin-Heidelberg, MICCAI2005 (J.Duncan and G.Gerig eds.), 657-665, 2005.

Prof. Dr. Andrei Bogatyrev
Inst. Num. Math. RAS
ul. Gubkina, 8
Moscow GSP-1
119991 RUSSIA
e-mail: gourmet@inm.ras.ru