ISSN 0001-4346, Mathematical Notes, 2012, Vol. 91, No. 1, pp. 135–138. © Pleiades Publishing, Ltd., 2012.
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SHORT COMMUNICATIONS

Elementary Construction of Jenkins–Strebel Differentials

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Received September 14, 2010; in final form, January 21, 2011

DOI: 10.1134/S0001434612010130

Keywords: *Jenkins–Strebel differential, Riemann surface, foliation, algebraic curve, reflection, intersection index.*

On a compact Riemann surface, the global behavior of trajectories of a regular quadratic differential is chaotic except for one case. If the critical graph of the foliation is compact, then its complement is a finite set of cylinders fibered into homotopic closed trajectories [1]. Such differentials arise when solving extremum problems of the geometric theory of functions [2], in the problem of decomposition of decorated moduli space of curves into cells (enumerated by ribbon graphs) [3], and in several other problems. Such differentials are now called Jenkins–Strebel differentials after the names of the scientists who proved their existence in the 1950–1960s [4]–[7]. A simpler proof was given later by Wolf [8]. All these results are pure existence theorems, and there are very few explicit constructions of such differentials. Several one-parameter families of Jenkins–Strebel differentials are explicitly described in [9], [10]. Of course, the Jenkins–Strebel differentials are not something extraordinary (they are dense in the space of quadratic differentials), but to verify whether their trajectories are closed is not a very simple problem.

Here we present an explicit multiparametric construction of Jenkins–Strebel differentials on real algebraic curves. Each squared real holomorphic differential of the 1st kind subjected to some explicit linear constraints is a Jenkins–Strebel differential.

We assume that an anticonformal involution $\overline{J}$ (reflection) acts on a compact Riemann surface X of genus g. The components of the set of fixed points of this involution are smooth closed curves [11] and are called *real ovals*. The reflection naturally acts on the $2g$-dimensional real space of the one-dimensional homology spaces of the surface and splits into the sum of spaces corresponding to the eigenvalues ±1 of the operator $\overline{J}$:

$$\mathbb{R}^{2g} \cong H_1(X,\mathbb{R}) = H_1^+(X,\mathbb{R}) \oplus H_1^-(X,\mathbb{R}), \qquad H_1^\pm(X,\mathbb{R}) := (I \pm \overline{J})H_1(X,\mathbb{R}). \tag{1}$$

The cycles $C = \overline{J}C$ of the space $H_1^+(X)$ are said to be *even*. Correspondingly, the cycles $C = -\overline{J}C$ of the space $H_1^-(X)$ are said to be *odd*. The spaces of even and odd cycles contain lattices of integer cycles

$$H_1^\pm(X,\mathbb{Z}) := H_1^\pm(X,\mathbb{R}) \cap H_1(X,\mathbb{Z})$$

of full rank. For example, the oriented real ovals of a curve are even integer cycles.

On the space $H_1(X,\mathbb{R})$, there is a nondegenerate skew bilinear form, i.e., the intersection index of the cycles. Since the mapping $\overline{J}$ changes the orientation at each intersection point of integer cycles, we have

$$\overline{J}C_1 \circ \overline{J}C_2 = -C_1 \circ C_2, \qquad C_1, C_2 \in H_1(X,\mathbb{R}). \tag{2}$$

This easily implies that the subspaces of even and odd cycles are Lagrangian (i.e., the restriction of the intersection form to them is zero) and their dimensions coincide and are equal to g.

The space $\Omega^1(X) \cong \mathbb{C}^g$ of holomorphic differentials on a curve contains the subspace of so-called real differentials $\Omega^1_{\mathbb{R}}(X) \cong \mathbb{R}^g$, which become the complex conjugate differentials $\overline{J}^*\eta = \overline{\eta}$, $\eta \in \Omega^1_{\mathbb{R}}(X)$,

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under reflection. The integrals of real differentials over even (resp., odd) cycles are real (resp., pure imaginary) numbers:

$$\int_C \xi = \int_{\pm\overline{J}C} \xi = \pm\int_C \overline{J}^*\xi = \pm\int_C \overline{\xi} = \pm\overline{\int_C \xi}.$$

Example. We assume that a curve X is given by the equation $P(x,y)=0$ with a real polynomial P. The curve admits the reflection $\overline{J}(x,y) := (\overline{x},\overline{y})$, and the real differentials have the form

$$Q(x,y)/P_y(x,y)\,dx$$

with an appropriate real polynomial Q.

Lemma 1. *The space $(H_1^-(X,\mathbb{R}))^* \cong \mathbb{R}^g$ of real linear functionals over odd cycles is canonically isomorphic to the following two spaces*:

(i) $H_1^+(X,\mathbb{R})$;

(ii) $\Omega^1_{\mathbb{R}}(X)$.

Proof. (i) In this case, the functional is defined by the intersection form. Since this form is nondegenerate, it follows that an even cycle for which all odd cycles are zero is itself zero.

(ii) In this case, the functional is defined by the formula

$$\langle \eta \mid C^-\rangle := i\int_{C^-}\eta, \qquad C^- \in H_1^-(X).$$

If all odd cycles for a real differential are zero, then all its periods are real. Such a differential is equal to zero. $\square$

Remark 1. Abelian differentials are usually normalized by using half of the canonical basis in homology spaces, namely, A-cycles or B-cycles. It follows from assertion (ii) of the lemma that even or odd cycles can be used as normalization cycles on a surface that admits reflections. Generalizing this observation, we show that, for the normalization of differentials, it is possible to use any Lagrangian subspace of dimension g in the homology spaces, i.e., on a curve there is a unique holomorphic differential with given periods on a basis in such a subspace.

We choose a basis $C_1, C_2, \dots, C_{2g}$ in the real homology space of the curve X so that its first g elements lie in the Lagrangian subspace. We do not assume that the basis is canonical or integer. The following bilinear Riemann relation holds:

$$0 \le \|\eta\|^2 = i\int_X \eta\wedge\overline{\eta} = -i\sum_{s,j=1}^{2g} F_{sj}\int_{C_s}\eta\overline{\int_{C_j}\eta}, \tag{3}$$

where the matrix F_{sj} is the inverse of the intersection matrix $C_s \circ C_j$. If

$$\int_{C_j}\eta = 0 \qquad \text{for} \quad j=1,\dots,g,$$

then the sum in the right-hand side contains only terms with $s,j>g$. But then we have $F_{sj}=0$. Indeed, the intersection matrix has a 2×2 block structure with zero $g\times g$ block at $(1,1)$. The inverse matrix also has a zero block of the same dimension at $(2,2)$. As we see, only the zero holomorphic differential has zero periods along all cycles in our Lagrangian subspace.

Remark 2. Comparing the two assertions of the lemma, we see that there is a one-to-one correspondence between even cycles C^+ and real differentials η on a curve according to the rule

$$i\int_C \eta = C^+ \circ C \qquad \text{for all} \quad C \in H_1^-(X,\mathbb{R}).$$

For even cycles C, this correspondence already does not hold, because it is not the Poincaré duality assignment.

Remark 3. Assume that there are k real ovals on a curve X. Their linear span in the homology space has dimension k if the curve with eliminated ovals does not split into components. Otherwise, this dimension is less by 1. We shall show that the corresponding subspace of real differentials generates Jenkins–Strebel foliations.

Theorem 1. *Suppose that the integral of a real holomorphic differential η over the odd cycle C^- is zero if the intersection index of C^- with any real oval is zero. Then the foliation $\eta^2 > 0$ is a Jenkins–Strebel foliation.*

Proof. We cut our surface along the real ovals. On the obtained surface with boundaries, the following function is well defined:

$$H(x) := \operatorname{Im} \int_*^x \eta, \qquad x \in X \setminus \{\text{real ovals}\}.$$

Indeed, if the closed path C does not intersect the real ovals, then

$$2 \operatorname{Im} \int_C \eta = \operatorname{Im} \int_{C - \overline{J}C} \eta = 0$$

by the assumption of the theorem, because

$$(C - \overline{J}C) \circ C^+ = 2C \circ C^+ = 0$$

for each real oval C^+. $\square$

This globally defined function is locally constant on the boundaries of the cut surface, and its level lines are leaves of the foliation $\eta^2 > 0$; see the figure.

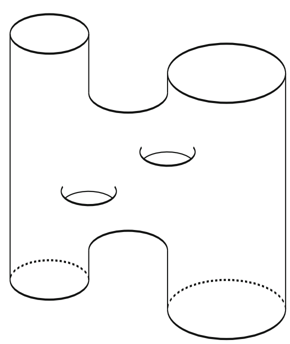

Figure: Function $H(x)$ as a function of height on the cut surface.

Example. We consider a hyperelliptic curve all of whose branch points are real except, possibly, for two points that are complex conjugate in this case. Its real ovals generate the entire space of even cycles, and hence the space annihilator is trivial. By the above theorem, any squared real holomorphic differential on such a curve is a Jenkins–Strebel differential. In the case of general real hyperelliptic curves, the bases in the lattices of even and odd integer cycles were considered in [12], and they permit explicitly describing the annihilator of all real ovals in the space of odd cycles.

ACKNOWLEDGMENTS

This work was supported by the Russian Foundation for Basic Research (grants nos. 10-01-00407 and 09-01-12160) and by the program of the Branch of Mathematics, Russian Academy of Sciences, “Contemporary Problems of Theoretical Mathematics”.

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