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Effective solution of the problem of the optimal stability polynomial

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Abstract. An effective method for finding the polynomial approximating the exponential function with order 3 at the origin and deviating from 0 by at most 1 on the longest interval of the real axis is put forward. This problem is reduced to the solution of four equations on a 4-dimensional moduli space of algebraic curves. A numerical realization of this method using summation of linear Poincaré series is described.

Bibliography: 19 titles.

§ 1. Introduction

The Runge–Kutta method has now been used for more than a century for numerical integration of systems of ordinary differential equations. About 50 years ago, in the development of n-stage explicit stable Runge–Kutta methods of the pth order of accuracy the following optimization problem for polynomials was put forward [1], [2].

Problem A. *Find a real polynomial* $R_n(x)$ *of degree* $\leqslant n$ *approximating the exponential function with order* $p \leqslant n$ *at the origin*: $R_n(x) = 1 + x + x^2/2! + \dots + x^p/p! + o(x^p)$, *such that its deviation* $\|R_n\|_E := \max_{x\in E} |R_n(x)|$ *is not larger than* 1 *on a possibly large interval* $E = [-L, 0]$, $L > 0$.

The solution of Problem A is known as the *optimal stability polynomial*. For $p = 1$ it can be expressed [3] in terms of the classical Chebyshev polynomials: $R_n(x) = T_n(1 + x/n^2)$. The Zolotarëv polynomials, which can be parametrically expressed in terms of elliptic functions, provide a solution of Problem A for $p = 2$ [4]. Many authors (Riha [5], Metzger, Lomax, Lebedev [6], van der Houwen, Medovikov, Abdulle [7], Verwer [8]) have pointed out that no closed analytic form is known for the solution $R_n(x)$ if $p > 2$ and have put forward various iterative methods for its numerical evaluation. The direct numerical optimization is extremely labour-consuming and in effect impossible for polynomials of large degree. The best iterative method known so far [6] requires 96 hours of calculation on a many-processor

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work-station for the solution of the problem for $p = 3$ and $n = 576$ (Lebedev and Medovikov).

It is nevertheless possible to find an analytic formula for the solution also for $p > 2$. The optimal stability polynomial turns out to satisfy the following definition for some $g \leqslant p - 1$.

We say that a real polynomial $P_n(x)$ is *g-extremal* if all its critical points, with the exception of g of them, are simple and the corresponding values are ± 1.

The theory of g-extremal polynomials is developed in [9]–[11]. These polynomials can be conveniently described by means of the following construction going back to Chebyshev. We associate with a real polynomial $P_n(x)$ the real hyperelliptic curve

$$M = M(\mathbf{e}) = \biggl\{(x,w) \in \mathbb{C}^2 : w^2 = \prod_{s=1}^{2g+2} (x - e_s)\biggr\}, \qquad \mathbf{e} := \{e_s\}_{s=1}^{2g+2}, \tag{1}$$

with branching divisor $\mathbf{e}$ equal to the odd-order zeros of the polynomial $P_n^2(x) - 1$. The genus of the curve M associated with a g-extremal polynomial is equal to g, the number of exceptional critical points of the polynomial counted with multiplicities [9]. The polynomial can be recovered up to the sign from its algebraic curve by the explicit formula

$$P_n(x) = \pm \cos\biggl(ni \int_{(e,0)}^{(x,w)} d\eta_M\biggr), \qquad x \in \mathbb{C}, \quad (x,w) \in M, \tag{2}$$

where $d\eta_M$ is a certain Abelian differential of the 3rd kind rigidly attached to M and the expression on the right-hand side is independent of the integration path on M, the double-valuedness of $w(x)$, and the branching point $e \in \mathbf{e}$ taken for the initial point of integration. The curves M generated by polynomials of degree n are not arbitrary: they satisfy Abel's equations

$$\int_{C_s} d\eta_M = 2\pi i \frac{m_s}{n}\,, \qquad s = 1, \dots, 2g, \quad m_s \in \mathbb{Z}, \tag{3}$$

where C_s is a basis in the lattice of integral 1-cycles on M. Half of these relations are a formal consequence of the mirror symmetry of the curve $\overline{J}(x,w) := (\overline{x}, \overline{w})$.

One can solve the problem of the optimal stability polynomial by choosing a suitable substitution (this is the Ansatz). One must start by 'guessing' the Ansatz, that is, determining the topological type of the real curve corresponding to the solution and the integers in Abel's equations. Next, using the input data of the problem, one makes up and solves a system of transcendental equations for the moduli of the associated curve. After that one recovers the solution by the explicit formula (2). The implementation of this scheme is the subject of the present paper.

The analytic approach to the solution of the problem of the optimal stability polynomial put forward here is numerically realized for approximation order $p = 3$. The solution of the above-mentioned problem for $n = 576$ by means of the algorithm thus developed now takes just several seconds of PC calculations.

§ 2. Properties of optimal stability polynomials

The explicit analytic representation of the solution of optimization problem A is based on a thorough investigation of the qualitative behaviour of this solution. The problem of the optimal stability polynomial is closely connected with the much better studied least deviation problem.

Problem A′. *Let E' be a bounded closed subinterval of the real axis. Minimize the norm $\|P_n\|_{E'}$ of a polynomial with prescribed r independent linear constraints on its coefficients.*

A solubility criterion for such problems with *Chebyshev* constraints is due to Bernstein.

Theorem 1 [12]. *Assume that each polynomial of degree $\leqslant n$ satisfying the homogeneous linear constraints of Problem A′ has at most $n-r$ zeros on E' with multiplicities taken into account. Then the solution $P_n(x)$ of the least deviation problem is unique and is characterized by the following property: $P_n(x)$ has an $(n+2-r)$-alternance on E', that is, at some $n+2-r$ points in the interval the polynomial takes the values $\pm\|P_n\|_{E'}$ with alternating signs.*

Theorem 2 [5], [9]. *The problem of the optimal stability polynomial is uniquely soluble. The polynomial*

$$R_n(x) = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^p}{p!} + O(x^{p+1}) \tag{4}$$

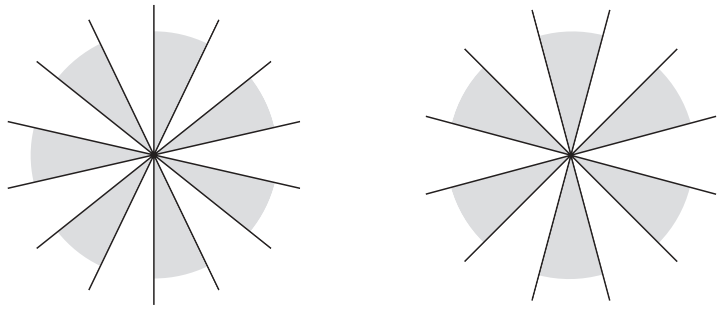
and the interval $E = [-L, 0]$ on which the deviation $\|R_n\|_E$ is equal to 1 solve Problem A if and only if this polynomial has an $(n+1-p)$-alternance on $E \setminus \{0\}$.

Proof. The closed ball $\{\|P_n\|_{[-l,0]} \leqslant 1\}$ in the space of polynomials of degree $\leqslant n$ contracts (linearly, but inhomogeneously) as $l > 0$ increases. In the limit as $l \to \infty$ it contains only constant polynomials, which for $p > 0$ do not satisfy the constraints (4). Hence there exist a longest interval $E := [-L, 0]$ and a polynomial $R_n(x)$ satisfying the constraints and with deviation 1 on E.

(1) We claim that $R_n(x)$ is at the same time a solution of Problem A′ with constraints (4) on the interval $E' = [-L, -\epsilon]$, where the positive quantity ϵ is smaller than each of the quantities 1, $L/2$, $1/\max|P_n''(x)|$ and the maximum is considered on the compact set $\{(P_n, x) : x \in [-L/2, 0]; \|P_n\|_{[-L,-L/2]} \leqslant 1; \deg P_n \leqslant n\}$. Assume that there exists a polynomial $P_n(x)$ with deviation less than 1 on E' satisfying the $r = p+1$ constraints (4). In view of the local increase of $P_n(x)$ in the neighbourhood of zero and the smallness of ϵ the deviation of P_n on E is 1. Since the value of $P_n(x)$ at the extreme point $x = -L$ is less than 1 in absolute value, E can be increased for the same norm of $P_n(x)$, in contradiction with the maximality of E.

The approximation of the exponential function by a polynomial at the origin with prescribed order p means the imposition of $r = p + 1$ linear inhomogeneous constraints (4) on its coefficients. A polynomial of degree $\leqslant n$ satisfying the homogeneous constraints has at most $n-r$ zeros on the closed interval E' because it has a zero of order r for $z = 0$. By Bernstein's theorem the least deviation polynomial $R_n(x)$ is unique and has an $(n+1-p)$-alternance on E'.

(2) Conversely, let $R_n(x)$ be a polynomial with an $(n+p-1)$-alternance on a half-open interval $[-L,0)$ on which it has deviation 1. By Bernstein's theorem $R_n(x)$ solves the least deviation problem with constraints (4) on the set $E'=[-L,-\epsilon]$ with sufficiently small $\epsilon>0$. The optimal stability polynomial has deviation 1 on E' under the same constraints. By the unique solubility of the least deviation problem R_n is the optimal stability polynomial.



(a) (b)

Figure 1. Order stars in the neighbourhood of zero for (a) p even and (b) p odd

The existence of an alternance enables one to find an estimate for the number of zeros of the optimal stability polynomial and its derivative on the stability set $E=[-L,0]$. Their precise number is described by the following result.

Lemma 1. *The solution* $R_n(x)$ *of optimization Problem* A *and its derivative* $\dfrac{dR_n(x)}{dx}$ *have only simple zeros, which lie in* E *and* $\mathbb{C}\setminus\mathbb{R}$ *in the following amounts*:

The number of zeros of R_n	E	$\mathbb{C}\setminus\mathbb{R}$
p even	$n-p$	p
p odd	$n-p+1$	$p-1$

The number of zeros of $\dfrac{dR_n}{dx}$	E	$\mathbb{C}\setminus\mathbb{R}$
p even	$n-p+1$	$p-2$
p odd	$n-p$	$p-1$

Proof (after Abdulle [7]). Assume that a real polynomial approximates the exponential function at the origin with precise order p: $P_n(x)-\exp(x)\asymp x^{p+1}$. Then it has at least $2[p/2]$ distinct complex zeros: the proof is based on the analysis of the topology of the order stars [2].

We consider two open subsets of the complex plane that are symmetric relative to the real axis. At points in the white subset $P_n(x)/\exp(x)$ is less that 1 in absolute value, in the black subset its absolute value is greater than 1. These two subsets, which are called the order stars, have the following easily verified properties [2]:

(a) the black and the white sets have precisely one unbounded component each;
(b) in the neighbourhood of the origin these subsets make up curvilinear sectors of angle $\pi/(p+1)$ and of alternating colours (see Fig. 1);
(c) each bounded component of the white set contains a zero of the polynomial (the maximum principle);

(d) the black set has no bounded components (they would contain poles of the polynomial) and is therefore connected.

Hence one concludes that an arbitrary component of the white set contains at most one sector: otherwise (d) fails. If the white component contains a sector lying strictly in the upper or the lower half-plane, then the entire component lies in this half-plane since the white subset is mirror-symmetric. In addition, such a component must be bounded: by (a) an unbounded component intersects both half-planes. We see from Fig. 1 that for even p there exist p white sectors disjoint from the real axis and for odd p there exist at least $p-1$ such sectors. Each of them lies in some bounded white component disjoint from the real axis and containing a zero of the polynomial $P_n(x)$ by (c). Correspondingly, our polynomial has at least $2[p/2]$ complex zeros and its derivative has at least $2[(p-1)/2]$ complex zeros.

One can say more about the position of the zeros of the optimal stability polynomial $R_n(x)$ and its derivative. Between two neighbouring points of the alternance there exists a zero of the polynomial, and each point of the alternance lying in the interior of E is a zero of its derivative. The interval between the origin and the extreme right point x_1 in the alternance contains either a zero of the polynomial (for $R_n(x_1)=-1$) or a zero of its derivative (for $R_n(x_1)=1$). For even p we have already found $(n-p)+p=n$ distinct zeros of R_n and for odd p we have found $(n-p)+(p-1)=n-1$ distinct zeros of dR_n/dx. Hence $R_n(x_1)=(-1)^p$ and the distribution of zeros is as required in the lemma.

Remark. We now list all the real zeros of the optimal stability polynomial $R_n(x)$ and its derivative again. Lying between pairs of neighbouring points in the alternance are $n-p$ zeros of the polynomial. In addition, the interval $(x_1, 0)$ contains a zero of the polynomial for p odd and a zero of its derivative for p even. The other real zeros of $dR_n(x)/dx$ are located at the $n-p$ points of the alternance lying in the interior of E. In particular, the left end-point of E is in the alternance.

§ 3. Chebyshev representation of solutions

On finding the curve M corresponding to the solution of the optimization problem we can recover the solution itself by the explicit formula (2). The complexity of the calculations by this formula in no way depends on the degree n of the solution, provided that one can effectively calculate the hyperelliptic integral.

3.1. The topological type of the associated curve. The branching points of the curve M associated with the polynomial $P_n(x)$ are even-order zeros of the polynomial $P_n^2(x)-1$. The total number $2g+2$ of the branching points and their number $2k$ on the real axis are topological invariants of the real curve. They determine the genus g of M (that is, the number of handles) and the number k of real ovals[1] (that is, contours fixed by the reflection $\overline{J}$ of the curve).

Theorem 3. *A real curve M associated with the optimal stability polynomial R_n contains precisely one real oval and has genus $g \leqslant p-1$.*

Proof. The end-points of the interval E are simple zeros of the polynomial $R_n^2(x)-1$; the $n-p$ points of the alternance lying in the interior of E are its double zeros.

[1]There exist precisely k real ovals for $k>0$; for $k=0$ their number is 1 or 2.

The inverse image $R_n^{-1}(\pm 1)$ has no other points on the real axis since otherwise the polynomial R_nR_n' would have zeros not described by Lemma 1. For example, the existence of a point $x \in R_n^{-1}(\pm 1)$ outside E means that the polynomial R_n or its derivative have a zero between x and E. The other cases can be considered in a similar fashion.

Hence there exist precisely $2p-2$ inverse images (taking multiplicities into account) of the points ± 1 outside the real axis. If these inverse images are simple, then M has genus $p-1$. If some of them are multiple, then g is smaller.

The natural conjecture that the genus of M is equal to $p-1$ holds for $p<4$.

Proposition. *For $p \leqslant 3$ the genus is $g = p-1$.*

Proof. If $p<3$, then there exists in the upper half plane at most one point in $R_n^{-1}(\pm 1)$ with multiplicities taken into account. If $p=3$, then each of the points -1 and $+1$ has $n-2$ inverse images (with multiplicities) on the real axis. Hence the upper half-plane contains precisely one simple inverse image of these points.

We see that for $p=1$ the solution of the optimization problem can be expressed in terms of the 0-extremal (Chebyshev) polynomial and for $p=2$ in terms of the 1-extremal (Zolotarëv) polynomial. In the rest of this paper we shall thoroughly study the case $p=3$ corresponding to the 2-extremal polynomial.

3.2. The moduli space. For $p=3$ the branching divisor of the curve M associated with the optimal stability polynomial consists of two real points and two pairs of complex conjugate points. The group $\mathfrak{A}_1^+$ of orientation-preserving affine motions of the real axis: $\mathbf{e} = \{e_s\}_{s=1}^6 \to A\mathbf{e}+B = \{Ae_s+B\}_{s=1}^6$, $A>0$, $B \in \mathbb{R}$, acts freely on such sets. We call the orbits of this action the *moduli space* $\mathcal{H}$. Each point in $\mathcal{H}$ defines a conformal class of (hyperelliptic) real curves of genus 2 with one oriented real oval and a distinguished point ∞_+ on it distinct from the branching points.

It is convenient to normalize the branching divisor $\mathbf{e}$ so that the left real branching point is at -1 and the right at 0. Then the branching divisor is completely determined by its two points in the open upper half-plane $\mathbb{H}$. In other words, the moduli space $\mathcal{H}$ is the 2-configuration space of the upper half-plane. It has real dimension 4 and fundamental group $\mathbb{Z}$ [9].

Normalized in the above-described way is the branching divisor $\mathbf{e}$ of the curve M associated with the polynomial $P_n(x) := R_n(Lx)$, which we shall call the *reduced optimal stability polynomial.*

3.3. Covering group. Uniformization of elements of the moduli space by means of Schottky groups produces the normal subgroup of the fundamental group of the punctured sphere $\widehat{\mathbb{C}} \setminus \mathbf{e}$ described below.

We join the branching points pairwise by cuts: an interval $\Lambda_0 := [-1,0]$, a simple smooth curve Λ_+ lying in $\mathbb{H}$ and connecting the branching points in the upper half-plane, and the mirror symmetric curve $\Lambda_- := \overline{\Lambda_+}$. The system of cuts $\Lambda := (\Lambda_+, \Lambda_0, \Lambda_-)$ defines a representation χ_Λ of $\pi_1(\widehat{\mathbb{C}} \setminus \mathbf{e}, \infty)$ into the abstract group $\mathfrak{G} := \langle G_+, G_0, G_- \mid G_+^2 = G_0^2 = G_-^2 = 1 \rangle$, the free product of three groups of order 2. We associate with a loop ρ intersecting transversally one after another the cuts $\Lambda_\star, \Lambda_\bullet, \dots, \Lambda_\circ$ (where the indices $\star, \bullet, \dots, \circ$ take the values $+, 0, -$) the element

$$\chi_\Lambda[\rho] := G_\star G_\bullet \cdots G_\circ$$

of this group. The representation into the discrete group $\mathfrak{G}$ cannot change after a continuous deformation of the cuts. Two arbitrary cuts of the upper half-plane having the same end-points are isotopic, therefore the representation χ_Λ and, in particular, its kernel, are independent of our choice of Λ_+.

3.4. Cycles on a Riemann surface. On each curve M in the moduli space $\mathcal{H}$ we distinguish four integral cycles. The contour C_0 goes counterclockwise along the banks of the interval $[-1, 0]$. The cycle C_+ encircles the pair of branching points in the upper half-plane (see Fig. 2(a), where we draw by bold lines the cuts $\Lambda_+, \Lambda_0, \Lambda_-$ connecting pairwise the branching points). The third cycle C_- is obtained from C_+ by a reflection and a change of orientation. The cycle C_1 goes along the banks of the cut connecting a branching point in the upper half-plane and the origin. By contrast with the first three, this cycle is not uniquely defined. The fundamental group of the moduli space acts naturally in the homology space of the curve (see [9], [11] for greater detail); as a result, an integer number of cycles C_+ can be added to C_1.

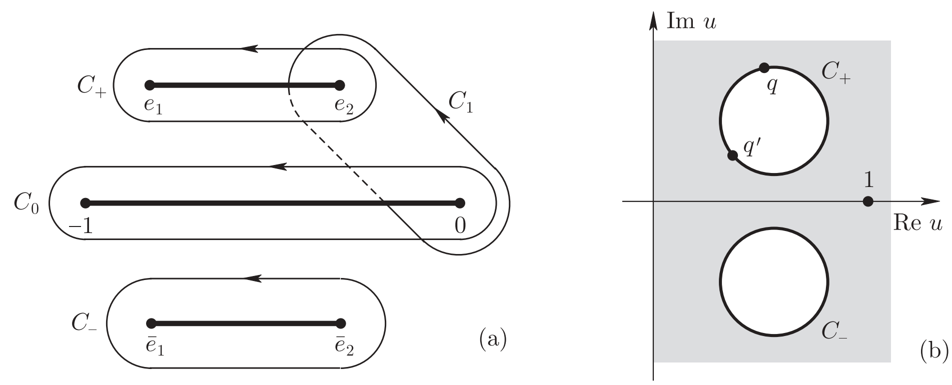


Figure 2. (a) The distinguished cycles on a curve M, an element of the moduli space $\mathcal{H}$. (b) The fundamental domain of the group $\mathfrak{G}$ associated with a point in the deformaton space $\mathcal{G}$

3.5. The associated differential. There exists on each curve (1) a unique Abelian differential with two simple poles at infinity, residue -1 at the point at infinity ∞_+ on the upper sheet, and purely imaginary periods. This normalization has a clear electrostatic analogy: one puts electric charges ∓ 1 at the points $\infty_\pm$; then the resulting distribution of the logarithmic potential on the Riemann surface M is the real part of the corresponding many-valued Abelian integral η_M. The differential associated with the curve M has the following coordinate representation:

$$d\eta_M = \left(x^g + \sum_{s=0}^{g-1} c_s x^s\right) \frac{dx}{w}\,. \tag{5}$$

It changes sign under the hyperelliptic involution $J(x,w) := (x,-w)$ of M. The real curve M admits the *reflection* (the anticonformal involution) $\overline{J}(x,w) := (\overline{x},\overline{w})$. One can verify that the reflection takes the associated differential $d\eta_M$ to $\overline{d\eta_M}$ [9]. Such differentials are said to be *real*; in our case this means that all the coefficients c_s in the representation (5) are real. It is easy to verify that the integrals of $d\eta_M$ over the cycle C and the reflected cycle $\overline{J}C$ are conjugate. In particular, the integral over each even cycle (such that $\overline{J}C = C$) vanishes.

3.6. Abel's equation. If the curve M is generated by a polynomial $P_n(x)$, then the associated differential has the following representation:

$$d\eta_M = n^{-1}\, d\log \widetilde{P}_n(x,w), \tag{6}$$

where $\widetilde{P}_n(x,w) := P_n(x) + \sqrt{P_n^2(x) - 1}$ is the (Akhiezer) meromorphic function on M with divisor $n(\infty_- - \infty_+)$. In fact, in view of the equality

$$\widetilde{P}_n(x,w)\widetilde{P}_n(x,-w) = 1,$$

the divisor of the Akhiezer function consists of the points ∞_- and ∞_+ covering infinity, with multiplicities n and $-n$. The differential (6) has only simple poles at infinity with residues ± 1 and purely imaginary periods; see [9] for details. The integral of (6) over each integral cycle can be expressed in terms of the argument of the Akhiezer function on this cycle and therefore belongs to the lattice $2\pi i\mathbb{Z}/n$. In particular,

$$\int_{C_1} d\eta_M \in 2\pi i\mathbb{Z}/n. \tag{7}$$

If the curve M is generated by the reduced optimal stability polynomial, then the integrals over the contours C_0 and C_+ can be precisely calculated. The polynomial

$$R_n(Lx) =: P_n(x) = \frac{1}{2}\left(\widetilde{P}_n(x,w) + \frac{1}{\widetilde{P}_n(x,w)}\right)$$

performs on $[-1,0]$ precisely $n-2$ oscillations between $+1$ and -1. As (x,w) moves along the contour C_0 on the Riemann surface the point $\widetilde{P}_n(x,w)$ obtained from $P_n(x)$ by the inverse Zhukovskiĭ map makes precisely $n-2$ counterclockwise circuits round the unit circle. Hence $\displaystyle\int_{C_0} d\eta_M = 2\pi i(n-2)/n$. The cycle $C_0 + C_+ + C_-$ contracts to the pole ∞_+ of the differential $d\eta_M$ the residue at which is -1. The integrals over C_+ and C_- are equal because the differential is real, therefore

$$\int_{C_+} d\eta_M = 2\pi i n^{-1}. \tag{8}$$

We shall see below that equations (7) and (8) ensure that each integral of $d\eta_M$ over an integral cycle lies in $2\pi i\mathbb{Z}/n$. Hence $\widetilde{P}_n(x,w) := \exp n \displaystyle\int_{(0,0)}^{(x,w)} d\eta_M$

is a single-valued function on the Riemann surface M. This is in effect another statement of Abel's criterion [13] for the existence of a function on M with divisor $n(\infty_- - \infty_+)$. We shall call (7), (8) *Abel's equations*. The polynomial P_n generating the curve M can be obtained from the Akhiezer function $\widetilde{P}_n$ by means of the Zhukovskiĭ map and can be calculated up to the sign by the explicit formula

$$P_n(x) = \cos\biggl(ni\int_{(0,0)}^{(x,w)} d\eta_M\biggr), \qquad x \in \mathbb{C}, \quad (x,w) \in M. \tag{9}$$

3.7. Equations of the moduli space. In the neighbourhood of the origin the reduced optimal stability polynomial approximates the function $\exp(Lx)$ with third order:

$$P_n(x) = 1 + Lx + \frac{(Lx)^2}{2} + \frac{(Lx)^3}{6} + O(x^4). \tag{10}$$

Eliminating straight away the unknown quantity $L = \dfrac{dP_n}{dx}(0)$ we obtain

$$\frac{d^2P_n}{dx^2}(0) = \biggl(\frac{dP_n}{dx}(0)\biggr)^2, \qquad \frac{d^3P_n}{dx^3}(0) = \biggl(\frac{dP_n}{dx}(0)\biggr)^3. \tag{11}$$

Theorem 4. *In the moduli space $\mathcal{H}$ the system of four equations* (7), (8), (11) *has a unique solution M. The function* (9) *at this point M is the reduced optimal stability polynomial.*

Proof. We have already carried out the proof in one direction: the curve associated with the reduced optimal stability polynomial satisfies Abel's equations as well as the constraints (11). We now claim that a curve satisfying these four equations gives rise to a solution of optimization Problem A.

The four cycles C_1, C_+, and their reflections $\overline{J}C_1$, $\overline{J}C_+$ form a (non-canonical) basis in the lattice of integer 1-homology of the compact curve M; their intersection indices make up an integer matrix with determinant ± 1. The integral of $d\eta_M$ over each integral cycle on M lies in $2\pi i\mathbb{Z}/n$ (in view of Abel's equations (7), (8) and since the differential is real). Hence formula (9) determines a single-valued meromorphic function on M. This function has singularities at the same points as the differential (that is, at infinity) and is invariant under the involution J, which changes the sign of the associated differential. Hence this function is a polynomial of x. Since $d\eta_M$ is real, it follows easily that so also is the polynomial $P_n(x)$. For x ranging from 0 to -1 the argument of the cosine function in (9) remains real and varies continuously from 0 to $-(n-2)\pi$ since $\displaystyle\int_{C_0} d\eta_M = 2\pi i(n-2)/n$ by Abel's second equation. Accordingly, the deviation of $P_n(x)$ on $[-1,0]$ is 1 and the polynomial itself possesses an $(n-2)$-alternance on the interval $[-1,0)$. We set $L := \dfrac{dP_n}{dx}(0) > 0$; then by the constraints (11) the 4-jet of the polynomial $P_n(x)$ has the form (10). By the criterion of Theorem 2, $P_n(x/L)$ is the optimal stability polynomial. Since the latter must be unique, our system of four equations is uniquely soluble in the moduli space $\mathcal{H}$.

§ 4. Schottky model

For an effective treatment of Riemann surfaces we shall uniformize the curves M by means of Schottky groups; then the moduli space will be represented as the deformation space of some Kleinian group. The quantities participating in our system of 4 equations and defined on the moduli space can be effectively calculated as the sums over the group of linear Poincaré series. A uniformization for which these series converge in the entire moduli space is described in [10]. We describe below another uniformization, for which the Poincaré series converge better in a neighbourhood of a solution, particularly for large n. Unfortunately, using this approach one can only ensure that the series converges in a (fairly large) piece of the moduli space.

4.1. The deformation space. A linear fractional transformation of order 2 with fixed points $c \pm r$ has the following form:

$$G_+(u) := G_+u = c + \frac{r^2}{u-c}. \tag{12}$$

Definition. By the *deformation space* $\mathcal{G}$ we shall mean the set of transformations (12) taking the interior of some simple smooth closed curve C_+ lying entirely in the open first coordinate sector onto the exterior of this curve (see Fig. 2(b)).

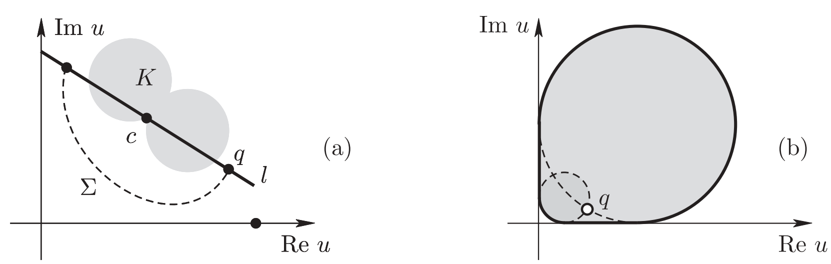


Figure 3. (a) The selection of the contour C_+. (b) A two-dimensional section of the classical part of the deformation space

One can parametrize the deformation space by the complex coefficients c and r^2 of the transformation G_+ or by unordered pairs q, q' of fixed points of it, which must lie in the first quadrant. We now describe explicitly the range of the moduli in the space $\mathbb{C}^2$.

Theorem 5. *The space $\mathcal{G}$ is defined by the following system of inequalities*:

$$|r| > 0, \qquad \operatorname{Re} c > |\operatorname{Re} r|, \qquad \operatorname{Im} c > |\operatorname{Im} r|, \qquad \operatorname{Im}(c^2 - r^2) > |r|^2. \tag{13}$$

Proof. (1) Let c, r^2 be the moduli defining a point in $\mathcal{G}$. The lower half-plane $-\mathbb{H}$ and the left half-plane $i\,\mathbb{H}$ lie outside C_+. Their $G_+(u)$-images lie inside C_+, that is, in the interior of the first quadrant of the u-plane. The image of the lower half-plane is the disc with centre at $c - r^2/(c-\overline{c})$ and of radius $|r|^2/|c-\overline{c}|$; the left half-plane is mapped onto the disc with centre at $c - r^2/(c + \overline{c})$ and with radius $|r|^2/|c + \overline{c}|$. A disc lies in the first quadrant if and only if both real and imaginary parts of its centre are larger than its radius. Hence, bearing in mind that $c = G_+(\infty)$ lies in the first quadrant one easily obtains inequalities (13).

(2) We shall demonstrate how one selects the contour C_+ required in the definition in the case when the moduli c and r^2 satisfy inequalities (13). Then the union of discs $K := G_+(-\mathbb{H}) \cup G_+(i\,\mathbb{H})$ lies in the interior of the first quadrant and contains no fixed points of the transformation $G_+(u)$. The line l passing through the points $c \pm r$ intersects the closure of K in a segment since the point c lies at the boundary of both discs. One can connect the fixed points $c \pm r$ of $G_+(u)$ by a smooth simple curve Σ lying in the first quadrant, to one side of l, and disjoint from the closure of K (see Fig. 3(a)). We take for the contour C_+ the union of Σ and $G_+(\Sigma)$; then we obtain a smooth closed curve without self-intersections that lies entirely in the first quadrant; the action of G_+ interchanges its interior and exterior. In fact, the map G_+ interchanges the half-planes to the left and to the right of l, while the doubly connected domain that is the complement of K with respect to the first quadrant is invariant.

Remark. It is not always possible to take a circle for C_+. If this is possible, then we say that the corresponding point in the deformation space is *classical*: the reasons will soon be clear. We now explain the relation between the classical and the non-classical parts of the space $\mathcal{G}$.

There exist precisely two circles passing through a fixed point q in the first quadrant and tangent to its sides. The interior of the convex hull of these circles (see Fig. 3(b)), minus q itself, is the locus of points q' such that q and q' are fixed points of a classical element G_+ of the deformation space. Corresponding to the deformation space in this figure are points $q' \neq q$ in the first quadrant lying inside the circle with centre at $x + 2y + i(y + 2x)$ and of radius $2x + 2y$, where we set $q = x + iy$. The two circles determined above by q are tangent to this circle from inside. Fixing one fixed point q one defines a two-dimensional section of the deformation space. The classical part takes on the average 82% of this section; its precise share depends on the argument of q.

4.2. The moduli space and the deformation space. A point G_+ in the deformation space generates a group $\mathfrak{G}$ on three generators: G_+u, $G_0u := -u$, $G_-u := \overline{G_+\overline{u}}$. By Klein's combination theorem [14] this group acts discontinuously in some subdomain $\mathcal{D} = \mathcal{D}(\mathfrak{G})$ of the sphere and all the relations in the group are consequences of three: $G_+^2 = G_0^2 = G_-^2 = 1$. The fundamental domain of this group is triply connected; it is bounded by the imaginary axis, the curve C_+ and its reflection $C_- := \overline{C_+}$. The Schottky group $\mathfrak{S}$ of genus 2 with generators $S_\pm := G_\pm G_0$ is a subgroup of $\mathfrak{G}$ of index 2. This Schottky group is *classical*, that is, with fundamental domain bounded by circles, if the generating point in the deformation space is classical.

The quotient variety by the Schottky group admits a hyperelliptic involution $J := G_0$ (with 6 fixed points $0, \infty, c \pm r, \overline{c} \pm \overline{r}$) and the reflection $\overline{J}u := \overline{u}$, so that it is a real hyperelliptic curve of genus 2. Selecting the natural orientation on the real curve and distinguishing a point '∞_+' $:= 1$ on it we obtain an element of the moduli space $\mathcal{H}$.

We now describe explicitly the map from the deformation space into the moduli space. The quotient variety by $\mathfrak{G}$ is the Riemann sphere. We normalize the natural projection $x(u)$ of the discontinuity set $\mathcal{D}$ of the group onto $\widehat{\mathbb{C}}$ so that it takes the points $0, 1, \infty$ to $0, \infty, -1$, respectively. Such a projection is unique and commutes with complex conjugation: $x(\overline{u}) = \overline{x(u)}$. Projecting the 6 fixed points of the hyperelliptic involution J we obtain the normalized branching divisor $\mathbf{e}$, the element of the moduli space $\mathcal{H}$ associated with the element G_+ of the deformation space $\mathcal{G}$.

The next result shows that the above construction gives one all the points in the moduli space.

Theorem 6. *The moduli space is isomorphic to the deformation space.*

Proof. (1) The *map* $\mathcal{G} \to \mathcal{H}$ *is monomorphic.* The projection $x(u)$ is an unbranched cover of the Riemann sphere punctured at the points in the branching divisor. The corresponding covering space is $\overset{\circ}{\mathcal{D}}$, the discontinuity domain, which is punctured at the fixed points of all the elliptic elements in $\mathfrak{G}$.

Let G_+^0 and $G_+^1 \in \mathcal{G}$ be elements producing the same branching divisor $\mathbf{e} \in \mathcal{H}$. We shall mark by superscripts 0 and 1 the corresponding groups, projections, and so on. The covering group $x^0(u)$ is the kernel of the representation χ_Λ in § 3.3 defined by the cut $\Lambda_+ := x^0(C_+^0)$ in the upper half-plane. The covering group $x^1(u)$ is defined by the cut $x^1(C_+^1)$ isotopic to the first cut. We have already observed that these groups coincide, therefore one can define a one-to-one map between the covering spaces

$$\check{f} = (x^1)^{-1}x^0 \colon \overset{\circ}{\mathcal{D}}{}^0 \to \overset{\circ}{\mathcal{D}}{}^1,$$

normalized by the equality $\check{f}(1) = 1$. This map is equivariant with respect to the covering transformation groups: $\check{f}G_\pm^0 = G_\pm^1\check{f}$, $\check{f}G_0 = G_0\check{f}$. In view of the equivariance, one can define the map $\check{f}$ by continuity at the punctures of $\mathcal{D}^0$. The discontinuity domain of the Schottky group is in the class O_{AD} (that is, each analytic function in this domain with finite Dirichlet integral is constant) and a univalent function in the discontinuity domain is linear fractional [15]. The points 0 and ∞ are fixed by $\check{f}$ alongside 1: either projection $x(u) = x^0(u)$ and $x^1(u)$ takes the paths $[0,1]$ and $[1,\infty]$ to the intervals $[0,\infty]$ and $[\infty,-1]$, respectively. Hence $\check{f}$ is the identity map. By the equivariance condition $G_+^0 = G_+^1$.

2. *The map* $\mathcal{G} \to \mathcal{H}$ *is epimorphic.* Consider an arbitrary element G_+ of the deformation space mapped into a divisor $\mathbf{e}$ in the moduli space. Let f be a plane diffeomorphism commuting with complex conjugation and taking $\mathbf{e}$ to a fixed divisor $\mathbf{e}'$. We shall also additionally assume that f is conformal in the neighbourhood of the points in $\mathbf{e}$. One can lift the Beltrami differential $\mu(x)\overline{dx}/dx := (f_{\overline{x}}\,\overline{dx})/(f_x dx)$ to the discontinuity domain $\mathcal{D}$ of $\mathfrak{G}$ by means of the branched cover $x(u)$. The new Beltrami differential $\widetilde{\mu}(u)\overline{du}/du$, $\widetilde{\mu}(u) := \mu(x(u))\dfrac{\overline{dx/du}}{dx/du}$, is $\mathfrak{G}$-invariant. The limit set of the group has plane measure zero, therefore the coefficient $\widetilde{\mu}(u)$ defines an element of $L_\infty(\mathbb{C})$ that is smooth on the discontinuity set $\mathcal{D}$ of the group. There exists a unique quasiconformal homeomorphism $\widetilde{f}(u)$ of the Riemann sphere satisfying Beltrami's equation $\widetilde{f}_{\overline{u}} = \widetilde{\mu}\widetilde{f}_u$ and stabilizing the three points 0, 1, ∞.

It follows by the $\mathfrak{G}$-invariance of the Beltrami differential that the homeomorphisms $\widetilde{f}(u)$ and $\widetilde{f}(Gu)$ differ by a conformal motion of the Riemann sphere:

$$G^f\widetilde{f}=\widetilde{f}G, \qquad G\in\mathfrak{G}, \quad G^f\in \mathrm{PSL}_2(\mathbb{C}).$$

The element $G+^f$, which is called the quasiconformal deformation of the element G_+, has order 2 and takes the exterior of the smooth curve $\widetilde{f}C_+$ to its interior. The curve $\widetilde{f}C_+$ lies in the image of the first quadrant. We claim that each quadrant is invariant under $\widetilde{f}(u)$, which means that G^f_+ is an element of the deformation space $\mathcal{G}$. The mirror symmetry of the Beltrami coefficient $\widetilde{\mu}(\overline{u})=\overline{\widetilde{\mu}(u)}$ and of the normalization set $\{0,1,\infty\}$ shows that $\widetilde{f}(u)$ commutes with complex conjugation. Hence $\widetilde{f}\,\mathbb{R}=\mathbb{R}$. In view of the normalization, the deformation of G_0 is trivial, therefore for $u\in i\,\mathbb{R}$ we have the chain of equalities $\widetilde{f}(u)=\widetilde{f}(-\overline{u})=-\widetilde{f}(\overline{u})=-\overline{\widetilde{f}(u)}$. We see that the real and imaginary axes are invariant, and moreover, keep their orientations, therefore all the quadrants are $\widetilde{f}$-invariant.

It remains to show that the so-obtained element G^f_+ of the deformation space is taken to the divisor $\mathbf{e}'\in\mathcal{H}$. Assume that an element G^f_+ corresponds to a projection normalized as before: $x^f(u)\colon \mathcal{D}^f=\widetilde{f}\mathcal{D}\to\widehat{\mathbb{C}}$. Using the uniqueness of the normalized quasiconformal map with fixed Beltrami coefficient we can show that $x^f\widetilde{f}=fx$. The branching divisor corresponding to G^f_+ is x^f (*the fixed points of* $G^f_\pm, G^f_0$) $= x^f\widetilde{f}$ (*the fixed points of* $G_\pm, G_0$) $= fx$ (*the fixed points of* $G_\pm, G_0$) $=$ $f\mathbf{e}=\mathbf{e}'$.

Remark. The method of [10] enables one to prove that the local coordinate system (e_1,e_2) (the points in the normalized divisor $\mathbf{e}$ in the upper half-plane) in the moduli space and the coordinate system (c,r^2) in the deformation space are related by a biholomorphism.

4.3. Linear Poincaré series. The Schottky groups $\mathfrak{S}$ corresponding to the classical part of the deformation space satisfy the following Schottky criterion. *The fundamental domain $\mathcal{R}$ bounded by the four circles C_+, C_-, $-C_+$, $-C_-$ can be partitioned into triply connected domains ('pants') bounded by circles.* As is known, the Poincaré series converge in this case absolutely and uniformly on compact subsets of the discontinuity domain of the group [16]. We can now effectively construct various analytic objects invariant under the action of $\mathfrak{S}$.

The Abelian differential of the third kind $d\eta_{zz'}$ with poles at the points z, z' in the fundamental domain can be obtained by averaging a differential on the sphere [17] over the group $\mathfrak{S}$:

$$\begin{aligned} d\eta_{zz'} := \dot{\eta}_{zz'}(u)\,du &:= \sum_{S\in\mathfrak{S}}\left\{\frac{1}{Su-z}-\frac{1}{Su-z'}\right\}dS(u)\\ &= \sum_{S\in\mathfrak{S}}\left\{\frac{1}{u-Sz}-\frac{1}{u-Sz'}\right\}du. \end{aligned} \tag{14}$$

The termwise equality of the two sums is a consequence of the infinitesimal form of the cross ratio identity:

$$\frac{\frac{d}{du}S(u)(z-z')}{(Su-z)(Su-z')}=\frac{S^{-1}z-S^{-1}z'}{(u-S^{-1}z)(u-S^{-1}z')}\,, \qquad S\in\mathrm{PSL}_2(\mathbb{C}). \tag{15}$$

Differentiating (14) with respect to the parameter z, the position of the pole, we obtain Abelian differentials of the second kind:

$$d\omega_{mz} := \dot{\omega}_{mz}(u)\,du := D_z^m\,d\eta_{zy}(u) = m!\sum_{S\in\mathfrak{S}}(Su-z)^{-m-1}\,dS(u),\qquad m=1,2,\dots. \tag{16}$$

One can obtain both holomorphic differentials on $M=\mathcal{D}/\mathfrak{S}$ by placing the poles z, z' in the same orbit of the group $\mathfrak{S}$ and separating out in (14) a telescopic sum:

$$\begin{aligned} d\zeta_\pm := \dot{\zeta}_\pm(u)du := d\eta_{S_\pm y\,y} &= \sum_{S\in\mathfrak{S}|\langle S_\pm\rangle}\bigl\{(u-S\alpha_\pm)^{-1}-(u-S\beta_\pm)^{-1}\bigr\}\,du \\ &= \sum_{S\in\langle S_\pm\rangle|\mathfrak{S}}\bigl\{(Su-\alpha_\pm)^{-1}-(Su-\beta_\pm)^{-1}\bigr\}\,dS(u), \end{aligned} \tag{17}$$

where $\alpha_\pm$ is the attracting fixed point and $\beta_\pm$ the repelling fixed point of the linear fractional map $S_\pm$; the sum is taken over representatives of the cosets by the subgroup $\langle S_\pm\rangle\subset\mathfrak{S}$ generated by the element $S_\pm$. The independence of the terms in (17) of one's choice of the representatives of the cosets follows from cross ratio identity (15). Integrating the series (14)–(16) termwise over the counterclockwise oriented circles $C_\pm$ we define the normalization of the differentials under consideration:

$$\int_{C_\pm}d\eta_{zz'}=0,\quad \int_{C_\pm}d\omega_{mz}=0,\quad \int_{C_\pm}d\zeta_\pm=2\pi i,\quad \int_{C_\pm}d\zeta_\mp=0,\qquad z,z'\in\mathcal{R}. \tag{18}$$

Called Schottky functions [16], [17], the exponentials of the integrals of the series in (14) and (17) can be effectively calculated:

$$(u,u';z,z') := \exp\int_{u'}^{u}d\eta_{zz'} = \prod_{S\in\mathfrak{S}}\frac{u-Sz}{u-Sz'}:\frac{u'-Sz}{u'-Sz'}\,, \tag{19}$$

$$E_\pm(u) := \exp\int_\infty^u d\zeta_\pm = \prod_{S\in\mathfrak{S}|\langle S_\pm\rangle}\frac{u-S\alpha_\pm}{u-S\beta_\pm}\,. \tag{20}$$

Under the action of $\mathfrak{S}$ the Schottky functions are transformed in accordance with the well-known formulae:

$$(S_\pm u,u';z,z') = (u,u';z,z')\frac{E_\pm(z)}{E_\pm(z')}\,, \tag{21}$$

$$E_\circ(S_\bullet u) = E_\circ(u)E_{\circ\bullet},\qquad \circ,\bullet\in\{+,-\}, \tag{22}$$

where the constant $E_{\circ\bullet}$, the exponential of the period of the holomorphic differential, can also be effectively calculated:

$$E_{\circ\bullet} = E_{\bullet\circ} = \prod_{S\in\langle S_\bullet\rangle|\mathfrak{S}|\langle S_\circ\rangle}\frac{S\alpha_\circ-\alpha_\bullet}{S\alpha_\circ-\beta_\bullet}:\frac{S\beta_\circ-\alpha_\bullet}{S\beta_\circ-\beta_\bullet}\,. \tag{23}$$

The product here is taken over two-sided cosets by the group $\mathfrak{S}$, and when the indices $\circ$ and $\bullet$ coincide, then the coefficient $0/\infty$ corresponding to $S=1$ must be replaced by the dilatation $\lambda_\circ := \dot{S}_\circ(\alpha_\circ)$.

4.4. Organization of the calculations. For the calculation of sums and products over a Schottky group or over its cosets by the subgroups $\langle S_+\rangle$, $\langle S_-\rangle$ we consider the Cayley graph of $\mathfrak{S}$. We put the elements of the group at vertices of an (infinite) tree and join each vertex S to the other four vertices $S_\pm S$, $S_\pm^{-1}S$. The tree is partitioned in the natural way into the levels with fixed norm $|S|$, the length of an irreducible factorization of an element S into a product of the generators. For instance, $|S_+^{-1}S_-S_+|=3$.

Consider the calculation of the function $\dot\eta_{zz'}(u):=d\eta_{zz'}/du$. It can be represented as two distinct series over the Schottky group: we have these series on the right- and left-hand sides of (14). For the calculation of a term of the right-hand series, for instance, at a vertex S_+S of the Cayley graph, we go one level down the tree and take the values $S(z)$, $S(z')$ stored at the vertex S. We put the quantities $S_+S(z)$, $S_+S(z')$ at the vertex S_+S under consideration and use them for the calculation of the term of the series corresponding to this vertex. This scheme is particularly efficient when one needs to calculate the values of the same series $\dot\eta_{zz'}$ for several values of the independent variable u. On the other hand if one needs to calculate the values of several series $\dot\eta_{zz'}$ at some point u, then one must store at the vertices of the tree the quantities $S(u)$, $\dot S(u)$, and must use for the calculations the left-hand series in (14). In either case for the summation of the terms of the series located at the same level of the Cayley graph one requires only data from the previous level.

Of course, one carries out the actual calculations over a finite subtree of the Cayley graph of a group. One can take, for instance, a tree of finite height; then one has a priori estimates for the error of the calculations. Practice shows that there is no sense in spending resources on the summation of small terms of the series until one has taken into account the larger ones. Hence it is more economical to take another subtree, which is determined in the process of calculation. Namely, if a term at a node S is less than some prescribed accuracy ε, then one need not carry out further calculations for the tree starting from this vertex. There exist estimates showing that the sum of the series over this infinite subtree has the same order ε as the term S at its root. With calculations organized in this way one knows the error only after the end of the process (a posteriori error estimate).

One can reduce the resources required for the calculation of Poincaré series by making use of the two involutions J, $\overline{J}$ on the group $\mathfrak{S}$. They are generated by the symmetries of M and commute: $JS(u):=-S(-u)$, $\overline{J}S(u):=\overline{S(\overline{u})}$, $S\in\mathfrak{S}$. The involution J is well defined on the cosets $\mathfrak{S}|\langle S_+\rangle$ and $\mathfrak{S}|\langle S_-\rangle$ of the group, while $\overline{J}$ takes the former cosets to the latter and conversely. One can group terms of Poincaré series mutually corresponding under either of the involutions. For instance, for $u\in\mathbb{R}$ we have

$$\dot\eta_{-11}(u)=\frac{-2}{u^2-1}+\sum_{1\neq S\in\mathfrak{S}/\sim}2\operatorname{Re}\biggl[\frac{1}{u-S(-1)}-\frac{1}{u-S(1)}+\frac{1}{u+S(1)}-\frac{1}{u+S(-1)}\biggr],$$

where the elements $S\neq1$ of $\mathfrak{S}$ are divided into classes $S\sim JS\sim\overline{J}S\sim\overline{J}JS$;

$$\dot\zeta_+(u)+\dot\zeta_-(u)=2\sum_{S\in\mathfrak{S}|\langle S_+\rangle}\operatorname{Re}\bigl[(u-S(\alpha_+))^{-1}-(u-S(\beta_+))^{-1}\bigr].$$

In particular, it follows from these equalities that the meromorphic differential $d\eta_{-11}$ and the holomorphic differential $d\zeta_+ + d\zeta_-$ are real. Besides the reduction of computations, with the use of such transformations one can often stay within real arithmetic.

4.5. Representation of functions. A non-trivial meromorphic function on the orbit variety of the group $\mathfrak{S}$ can be expressed in terms of Schottky functions.

Lemma 2 [10]. *Let $F(u)$ be an automorphic function with divisor*

$$\sum_{s=1}^{\deg F} (z_s - z'_s)$$

in the fundamental domain of the group $\mathfrak{S}$. Then the following representation holds:

$$F(u) = \text{const} \cdot \prod_{s=1}^{\deg F} (u, *; z_s, z'_s) E_+^{m_+}(u) E_-^{m_-}(u),$$

where $m_\pm \in \mathbb{Z}$ is the increment of $(2\pi i)^{-1} \log F(u)$ over the cycle $C_\pm$ and the position of the point $*$ only affects the constant on the right-hand side of this formula.

Remark. The automorphy condition, that is, the invariance of the function $F(u)$ under the action of $\mathfrak{S}$ gives one compatibility relations between the divisor and the integers $m_\pm$, which are easy to deduce from the transformation rules for Schottky functions (21), (22). These conditions are equivalent to Abel's criterion for the divisor of a meromorphic function.

Example 1. We calculate the projection $x(u)$ of the discontinuity domain of the Kleinian group $\mathfrak{G}$ onto the Riemann sphere normalized, as before, by the condition $(0, 1, \infty) \to (0, \infty, -1)$. The function $x(u)$ has a double zero at $u = 0$ and simple poles at $u = \pm 1$; in addition, the increment of the argument of $x(u)$ along the boundary circles $C_\pm$ is 0. Hence

$$dx(u)/x(u) = d\eta_{01}(u) + d\eta_{0-1}(u) = d\eta_{01}(u) + d\eta_{01}(-u)$$

and the projection has the following representation:

$$x(u) = -(u, \infty; 0, 1)(-u, \infty; 0, 1). \tag{24}$$

Example 2. Assuming that Abel's equations are fulfilled we calculate the polynomial P_n at the corresponding point in the deformation space by formula (9). The associated differential $d\eta_M$ on the curve $M := \mathcal{D}/\mathfrak{S}$ is real, therefore both integrals over the circles $C_\pm$ are equal to $2\pi i/n$ by Abel's second equation (8). This differential has simple poles at the points ± 1 with residues ∓ 1, therefore it has the following form:

$$d\eta^* = d\eta_{-11} + \frac{1}{n}(d\zeta_+ + d\zeta_-). \tag{25}$$

The Akhiezer function on M has the following form as a function of the uniformizing variable u:

$$\widetilde{P}_n(x,w) := \widetilde{T}(u) = \exp\biggl(n\int_0^u d\eta^*\biggr) = (u,0;-1,1)^n \frac{E_+(u)}{E_+(0)}\,\frac{E_-(u)}{E_-(0)}\,. \tag{26}$$

Finally, the 2-extremal polynomial $P_n(x)$ can be obtained from (26) by means of the Zhukovskiĭ map:

$$P_n(x) := T(u) = \frac{1}{2}\biggl(\widetilde{T}(u) + \frac{1}{\widetilde{T}(u)}\biggr). \tag{27}$$

§ 5. Equations on the deformation space

Formulae (24), (26), (27) in the previous section give one an effective parametric representation for the reduced optimal stability polynomial, provided that Abel's equations and the constraints are fulfilled at some point in the classical part of $\mathcal{G}$. We proceed to the derivation of these equations on the deformation space.

5.1. Abel's equations.

Lemma 3. *In the classical part of the deformation space $\mathcal{G}$ Abel's equations* (7), (8) *are equivalent to the single complex condition*

$$E_+^{2n}(1) = E_{++}E_{+-}. \tag{28}$$

Proof. (1) If Abel's second equation (8) holds, then in the Schottky model the differential $d\eta_M$ associated with the curve has the form (25). Equation (7) is equivalent to the condition $\exp\biggl(n\int_{C_1} d\eta_M\biggr) = 1$. Using successively Riemann's relation, the transformation rules for Schottky functions (22), and the oddness of the holomorphic differentials $d\zeta_+(-u) = -d\zeta_+(u)$ we obtain the sequence of equalities

$$\begin{aligned}\exp\biggl(n\int_u^{S_+u} d\eta^*\biggr) &= \exp\biggl(n\int_1^{-1} d\eta_{S_+u\,u} + \int_u^{S_+u}(d\zeta_+ + d\zeta_-)\biggr)\\ &= \biggl(\frac{E_+(-1)}{E_+(1)}\biggr)^n E_{++}E_{+-} = \frac{E_{++}E_{+-}}{(E_+(1))^{2n}}\,.\end{aligned}$$

(2) Going backwards along the above chain of equalities, by condition (28) we obtain $\int_{C_1} d\eta^* \in 2\pi i\mathbb{Z}/n$. The normalization (18) of the Abelian differentials yields $\int_{C_+} d\eta^* = \dfrac{2\pi i}{n}$. It remains to demonstrate that $d\eta^*$ is the associated differential on the curve $M = \mathcal{D}/\mathfrak{S}$. The differential $d\eta^*$ is real, so that the integrals over the reflected cycles $\overline{J}C_1$, $\overline{J}C_+$ on M are also purely imaginary. Considered on the orbit space of the group the differential $d\eta^*$ has simple poles at the distinguished points '$\infty_\pm$' $:= \pm 1$ with residues ∓ 1 and purely imaginary periods, therefore $d\eta^* = d\eta_M$.

Remarks. (1) Condition (28) is equivalent to Abel's equations on the entire space $\mathcal{G}$, provided that one defines the quantities involved in this condition without the use of Poincaré series.

(2) Condition (28) is equivalent to the automorphy of (26).

5.2. Constraints. If Abel's equations are fulfilled, then the function (9) is a polynomial, which in the Schottky model can be defined parametrically by formulae (24), (26), (27). We now find the first three derivatives of $P_n(x)$ at the origin; to this end we consider the jets of the functions $T(u) := P_n(x)$ and $x(u)$:

$$\begin{aligned} T(u) &:= \cosh\biggl(\int_0^u n\,d\eta^*\biggr) =: 1 + T_2u^2 + T_4u^4 + T_6u^6 + \cdots, \\ x(u) &= -\exp\biggl(-\int_\infty^u d\eta_{10} + d\eta_{-10}\biggr) =: x_2u^2 + x_4u^4 + x_6u^6 + \cdots. \end{aligned} \tag{29}$$

For the derivatives of $P_n(x)$ at the origin we have the formulae

$$\begin{aligned} \frac{dP_n}{dx}(0) &= \frac{T_2}{x_2}, \\ \frac{d^2P_n}{dx^2}(0) &= 2\frac{T_4x_2 - T_2x_4}{x_2^3}, \\ \frac{d^3P_n}{dx^3}(0) &= 6\frac{2(T_2x_4 - T_4x_2)x_4 + (T_6x_2 - T_2x_6)x_2}{x_2^5}. \end{aligned} \tag{30}$$

After substituting the expressions (30) the constraints (11) depend only on the projective jet $x_2 : x_4 : x_6$ of $x(u)$. They survive the dilation of the independent variable x.

5.3. The jet of $T(u)$. One calculates the coefficients η_l^* of the Taylor expansion

$$\frac{d\eta^*}{du} = \eta_0^* + \eta_2^*u^2 + \eta_4^*u^4 + \cdots$$

with the use of the convergent series

$$\begin{aligned} \eta_l^* = \frac{1}{l!}D_u^l\dot\eta^*(u)\big|_{u=0} &= \sum_{S\in\mathfrak{S}}\bigl((S(1))^{-l-1} - (S(-1))^{-l-1}\bigr) \\ &+ \frac{2}{n}\sum_{S\in\mathfrak{S}|\langle S_+\rangle}\operatorname{Re}\bigl[(S(\beta_+))^{-l-1} - (S(\alpha_+))^{-l-1}\bigr], \qquad l = 0, 2, 4, \dots. \end{aligned}$$

The first three non-trivial coefficients in the expansion of

$$T(u) := \cosh\biggl(n\int_0^u d\eta^*\biggr)$$

in powers of u are as follows:

$$\begin{aligned} T_2 &= \frac{(n\eta_0^*)^2}{2}, \\ T_4 &= \frac{(n\eta_0^*)(n\eta_2^*)}{3} + \frac{(n\eta_0^*)^4}{24}, \\ T_6 &= \frac{(n\eta_0^*)(n\eta_4^*)}{5} + \frac{(n\eta_2^*)^2}{18} + \frac{(n\eta_0^*)^3(n\eta_2^*)}{18} + \frac{(n\eta_0^*)^6}{720}. \end{aligned} \tag{31}$$

5.4. The projective jet of $x(u)$. Comparing the coefficients of distinct powers of u in the equality $dx(u) = (d\eta_{01}(u)+d\eta_{01}(-u))x(u) =: 2(1/u+\eta_1 u+\eta_3 u^3+\cdots)x\,du$ we obtain

$$x_2 : x_4 : x_6 = 2 : 2\eta_1 : (\eta_1^2+\eta_3).$$

The coefficients η_l of the Taylor expansion of $\dot\eta_{01}(u)$ in the neighbourhood of the origin can be calculated by the formulae

$$\eta_l = \frac{1}{l!}D_u^l\biggl(\dot\eta_{01}(u)-\frac{1}{u}\biggr)\bigg|_{u=0} = 1+\sum_{1\neq S\in\mathfrak{S}}\bigl((S(1))^{-l-1}-(S(0))^{-l-1}\bigr), \quad l=1,3,\dots.$$

5.5. Variational theory. For an effective solution of our system of four equations (28), (11) on the deformation space one must find the derivatives of the quantities involved in these equations with respect to the moduli. Abel's equations in the form (28) contain a period of the integral of the form $d\eta^*$. The constraints (11) in the Schottky model are certain relations between Abelian integrals with fixed limits of integration. Indeed, using Riemann's relations one can transform the coefficients of the jets of the differentials $d\eta^*$ and $d\eta_{01}$ to the required form:

$$\begin{aligned} l!\eta_l^* &= D_u^l\dot\eta^*(u)\big|_{u=0} = D_u^{l+1}\int^u d\eta^*(u)\bigg|_{u=0} \\ &= \int_1^{-1} d\omega_{(l+1)0} + \frac{1}{n}\biggl[\int_w^{S_+w} d\omega_{(l+1)0} + \int_w^{S_-w} d\omega_{(l+1)0}\biggr]. \end{aligned}$$

The position of w is not important here. The differential $d\eta_{01}$ has a singularity at $u=0$, therefore the coefficients of its jet are a regularization of the divergent integral:

$$l!\eta_l = D_u^l\biggl(\dot\eta_{01}(u)-\frac{1}{u}\biggr)\bigg|_{u=0} = \lim_{u\to 0}\biggl[\int_1^0 d\omega_{(l+1)u} + \frac{l!}{(-u)^{l+1}}\biggr].$$

The normalized Abelian integrals with fixed limits of integration and their periods are functions of a point in the deformation space $\mathcal{G}$. A slight perturbation of the modules δc, δr^2 results in a small perturbation of the matrices $\widehat S_\pm\in\mathrm{PGL}_2(\mathbb{C})$ corresponding to the generators of $\mathfrak{S}$:

$$\widehat S_+ := \begin{Vmatrix} c & c^2-r^2 \\ 1 & c\end{Vmatrix}, \quad \delta\widehat S_+ := \begin{Vmatrix} 1 & 2c \\ 0 & 1\end{Vmatrix}\delta c - \begin{Vmatrix} 0 & 2r \\ 0 & 0\end{Vmatrix}\delta r + o, \quad o := o(|\delta c|+|\delta r|). \tag{32}$$

The matrix $\widehat S_-$ and its perturbation $\delta\widehat S_-$ are related to $\widehat S_+$ and $\delta\widehat S_+$ by complex conjugation.

Theorem 7 [18], [10]. *The following formulae describe the variations of definite Abelian integrals and their periods*:

$$\delta\int_v^{v'} d\eta = \frac{1}{2\pi i}\sum_{\bullet=+,-}\int_{C_\bullet}\dot\eta(u)\dot\eta_{vv'}(u)\,\mathrm{tr}[\mathcal{M}(u)\cdot\delta\widehat S_\bullet\cdot\widehat S_\bullet^{-1}]\,du + o, \tag{33}$$

$$\delta\int_{S_\pm w}^{w} d\eta = \frac{1}{2\pi i}\sum_{\bullet=+,-}\int_{C_\bullet}\dot\eta(u)\dot\zeta_\pm(u)\,\mathrm{tr}[\mathcal{M}(u)\cdot\delta\widehat S_\bullet\cdot\widehat S_\bullet^{-1}]\,du + o, \tag{34}$$

where the limits of integration v, v' *lie in the fundamental domain* $\mathcal{R}$ *of the Schottky group*; *all the objects on the right-hand sides of these equalities relate to the unperturbed group*; $d\eta(u) := \dot\eta(u)du$ *is any of the differentials* $d\eta_{zz'}$, $d\zeta_\pm$, $d\omega_{mz}$ *with normalization* (18); $\mathcal{M}(u) := (u,1)^t \cdot (-1,u) \in \mathrm{sl}_2(\mathbb{C})$ *is the Hejhal matrix*; $o := o(|\delta c| + |\delta r|)$.

Remarks. (1) In effect (33) is the well-known Hadamard's formula for variations of the Green's function. One can find similar variational formulae by other authors; see the references in [18].

(2) The above variational formulae hold (with obvious modifications) for arbitrary Schottky groups of arbitrary genus. We verified these formulae by numerical experiment.

Applying Theorem 7 we obtain derivatives of all the quantities participating in the equations on the deformation space:

$$\delta\int_{C_1} d\eta^* \approx -\frac{1}{2\pi i}\sum_{\bullet=+,-}\int_{C_\bullet}\dot\eta^*(u)\dot\zeta_+(u)\,\mathrm{tr}[\mathcal{M}(u)\cdot\delta\widehat{S}_\bullet\cdot\widehat{S}_\bullet^{-1}]\,du,$$

$$\delta\eta_l \approx \frac{1}{2\pi i\, l!}\sum_{\bullet=+,-}\int_{C_\bullet}\dot\omega_{(l+1)0}(u)\dot\eta_{10}(u)\,\mathrm{tr}[\mathcal{M}(u)\cdot\delta\widehat{S}_\bullet\cdot\widehat{S}_\bullet^{-1}]\,du, \qquad l=1,3,$$

$$\delta\eta_l^* \approx -\frac{1}{2\pi i\, l!}\sum_{\bullet=+,-}\int_{C_\bullet}\dot\omega_{(l+1)0}(u)\dot\eta^*(u)\,\mathrm{tr}[\mathcal{M}(u)\cdot\delta\widehat{S}_\bullet\cdot\widehat{S}_\bullet^{-1}]\,du, \quad l=0,2,4.$$

5.6. Hejhal's formulae. The direct calculation of the variations by formulae (33), (34) is fairly labour-consuming since quadrature formulae require the calculation of series at many points. However, there exists a way round allowing one to arrive at the result after calculating the series only at three points. We follow Hejhal [19], who evaluated the maps

$$\Xi(u)(du)^2 \overset{[\pm]}{\longmapsto} \int_{C_\pm}\Xi(u)\mathcal{M}(u)\,du \in \mathrm{sl}_2(\mathbb{C}) \tag{35}$$

from the space of (meromorphic) quadratic differentials on the curve, which are involved in our variational formulae, for the (relative) quadratic Poincaré theta-series.

Consider three holomorphic quadratic differentials on $M := \mathcal{D}/\mathfrak{S}$:

$$\Omega_0(u)(du)^2 := \sum_{S\in\mathfrak{S}}\frac{(dS(u))^2}{((Su)^2-\alpha_+^2)((Su)^2-\alpha_-^2)}\,, \tag{36}$$

$$\Omega_\pm(u)(du)^2 := \sum_{S\in\langle S_\pm\rangle|\mathfrak{S}}\frac{(dS(u))^2}{((Su)^2-\alpha_\pm^2)^2}\,. \tag{37}$$

The absolute convergence of these series on compact subsets of the discontinuity domain of the group is a consequence of the convergence of (14), (17). For instance, the quadratic Poincaré series (37) are formed by the squares of terms of the linear series (17). The maps (35) for these series can be calculated by termwise integration.

Lemma 4 (Hejhal's formulae [18], [19]). *The following formulae hold*:

$$\int_{C_\pm} \Omega_0(u)\mathcal{M}(u)\,du = \frac{i\pi}{\alpha_\pm(\alpha_\pm^2-\alpha_\mp^2)(1-\lambda_\pm)}\left(\left\|\begin{matrix}-\alpha_\pm & \alpha_\pm^2\\ -1 & \alpha_\pm\end{matrix}\right\| + \lambda_\pm\left\|\begin{matrix}\alpha_\pm & \alpha_\pm^2\\ -1 & -\alpha_\pm\end{matrix}\right\|\right),$$

$$\int_{C_\pm} \Omega_\pm(u)\mathcal{M}(u)\,du = \frac{i\pi}{2\alpha_\pm^2}\left\|\begin{matrix}0 & \alpha_\pm\\ \alpha_\pm^{-1} & 0\end{matrix}\right\|, \qquad \int_{C_\mp} \Omega_\pm(u)\mathcal{M}(u)\,du = 0,$$

here $\lambda_\pm := \dot S_\pm(\alpha_\pm)$ is the dilatation coefficient of the generator $S_\pm$ of the Schottky group.

One consequence of Hejhal's formulae is as follows.

Lemma 5. *The quadratic Poincaré series* (36), (37) *form a basis in the space of holomorphic quadratic differentials on the curve M in the Schottky model.*

Proof. The space of holomorphic quadratic differentials of a curve of genus g has dimension $3g-3$. We claim that the three differentials (36), (37) on a curve of genus $g=2$ are linearly independent. Consider three functionals over quadratic differentials: the quantities at the positions $(1,1)$ and $(1,2)$ for the Hejhal $[+]$ map (35) and the quantity at the position $(1,2)$ for the map $[-]$. The values of these functionals at the differentials $\Omega_0(du)^2$, $\Omega_+(du)^2$, $\Omega_-(du)^2$ form an upper triangular matrix with non-singular diagonal: $-i\pi/(\alpha_+^2-\alpha_-^2)$, $i\pi/(2\alpha_+)$, $i\pi/(2\alpha_-)$. Hence these differentials are linearly independent and form a basis in the space of quadratic differentials on the curve.

How can one calculate the Hejhal map at a meromorphic quadratic differential such as $d\eta^* d\zeta_+$ or $d\omega_{(l+1)0}d\eta_{10}$? One must subtract from it a quadratic Poincaré series with suitable singularities. There exist analogues of Hejhal's formulae for such series [18]. One expands the remaining holomorphic quadratic differential with respect to the basis (36), (37), for which the Hejhal map is described by explicit formulae.

§ 6. Numerical experiments

The system of four equations (28) and (11) with substituted quantities (30) has at most one solution in the classical part of the deformation space $\mathcal{G}$ for fixed degree n. The author developed software for finding this solution by Newton's method. The first approximation for a low degree n was found by the trial-and-error method. The solution with numerical accuracy of our system of four equations for fixed n can be used as an initial approximation in Newton's method for systems of degrees $n+1,\dots,n+5$ and even $n+50$ with large n. We solved the equation to within 10^{-13} for degrees up to $n=1001$; here all the solutions (c,r^2) occur in the classical part of the deformation space. We could see no tendency towards deceleration of the convergence of the Poincaré series with growth of n: $\operatorname{Re} c$, $\operatorname{Im} c \gg |r|$ (see Table 1).

We also calculated the length L of the real stability domain. For $p=1$, when the solution can be expressed in terms of the Chebyshev polynomial, $L=2n^2$. One sees from Table 1 that for $p=3$ the quantity L/n^2 stabilizes — this was also pointed out by other authors. We plot the graph of the reduced optimal stability polynomial of degree $n=31$ for $p=3$ in Fig. 4.

TABLE 1

n	c	r^2	L/n^2
27	$0.058294188 + i0.072379887$	$3.92447195D - 005 + i1.12862487D - 005$	0.498393
107	$0.014619780 + i0.018280095$	$2.48122506D - 006 + i7.45913680D - 007$	0.500954
157	$0.009961570 + i0.012458796$	$1.15218232D - 006 + i3.46919646D - 007$	0.501047
199	$0.007858553 + i0.009829401$	$7.17095951D - 007 + i2.16027548D - 007$	0.501078
251	$0.006230206 + i0.007793080$	$4.50726679D - 007 + i1.35825821D - 007$	0.501097
301	$0.005195168 + i0.006498569$	$3.13412277D - 007 + i9.44617098D - 008$	0.501106
401	$0.003899525 + i0.004877994$	$1.76582873D - 007 + i5.32302909D - 008$	0.501116
501	$0.003121143 + i0.003904348$	$1.13124444D - 007 + i3.41035348D - 008$	0.501120
576	$0.002714732 + i0.003395971$	$8.55824263D - 008 + i2.58013073D - 008$	0.501122
651	$0.002401967 + i0.003004732$	$6.69986418D - 008 + i2.01991310D - 008$	0.501124
751	$0.002082125 + i0.002604635$	$5.03438797D - 008 + i1.51782584D - 008$	0.501125
851	$0.001837454 + i0.002298568$	$3.92072592D - 008 + i1.18208166D - 008$	0.501126
951	$0.001644239 + i0.002056868$	$3.13952470D - 008 + i9.46561689D - 009$	0.501126
1001	$0.001562108 + i0.001954128$	$2.83371730D - 008 + i8.54364345D - 009$	0.501126

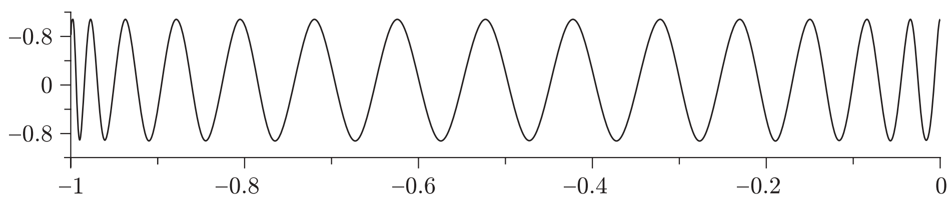


Figure 4. The reduced optimal stability polynomial for $p = 3$ and $n = 31$

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