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Effective approach to least deviation problems

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Abstract. A hierarchy of extremal polynomials described in terms of real hyperelliptic curves of genus $g \geqslant 0$ is constructed. These polynomials depend on g integer-valued and g continuous parameters. The classical Chebyshev polynomials are obtained for $g = 0$ and the Zolotarëv polynomials for $g = 1$.

Bibliography: 17 titles.

§ 1. Statement of the problem

Starting from Chebyshev, many authors (see the references in the monographs [1] and [2]) have considered extremal problems with constraints in the space of real polynomials

$$\left\{P_n(x) = \sum_{s=0}^{n} c_s x^s\right\} \cong \mathbb{R}^{n+1} \tag{1}$$

with uniform norm $\|P_n\|_E := \max_{x\in E} |P_n(x)|$, where E is a compact subset of the real axis. The following two examples relate to the optimization of numerical algorithms [3].

Problem A. *Let E be a system of several real intervals. Minimize the norm $\|P_n\|_E$ of a polynomial with fixed linear constraints imposed on its coefficients $c_0, c_1, \dots, c_n$.*

The zeros of a monic polynomial of least deviation can be used as iteration parameters in the inversion of symmetric matrices with spectrum in E. The Zolotarëv problem [1] corresponds to $E = [-1, 1]$ and several fixed leading coefficients of the polynomial.

Problem B. *Find the maximal interval $E = [0, t]$, $t > 0$, such that the unit ball $\{P_n : \|P_n\|_E \leqslant 1\}$ in the space* (1) *intersects the plane of codimension r consisting of the polynomials approximating* $\exp(-x)$ *at the origin with order $r - 1$.*

This problem (due to Lebedev) occurs in the construction of stable explicit integration schemes of $(r-1)$th order accuracy for stiff systems of ordinary differential equations.

The numerical solution of these and similar extremal problems for the degrees $n \approx 1000$ (of practical interest) is notorious for its complexity. The known

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algorithms of Remez [4], Lebedev [5], Peherstorfer–Schiefermayr [6], or convex programming methods are computationally expensive for the following reasons: the solution is sought by iterations in a space of large dimension (of order n) and the norm of a polynomial is a non-smooth function of its coefficients, which is difficult to evaluate.

The central point of the approach to least deviation problems put forward below is as follows: we do not seek the solution in the entire space of polynomials, but only on certain low-dimensional submanifolds of it. Criteria for the attainability of the minimum — for instance, Chebyshev's alternance principle [1] — suggest that the following situation must be typical. *Overwhelmingly, the critical points of the solution $T(x)$ are simple, lie in the set E, and the values of the polynomial at these points are* $\pm\|T(x)\|_E$. Polynomials of this kind are very special and form low-dimensional submanifolds of the space (1). The geometric interpretation is as follows. The solution of the extremal problem corresponds to the tangency between the (linear) submanifold of the space (1) describing the constraints of the problem for polynomials and the sphere consisting of polynomials of equal norm. The sphere corresponding to the uniform norm is not smooth: it is the boundary of a convex curvilinear polyhedron. Low-dimensional faces of the ball are its most 'protruding' parts, therefore it is no surprise that the various surfaces are more often tangent to these faces. It will be shown in § 2 that polynomials most commonly encountered among the solutions of least deviation problems have this form. This observation underlies the following definition.

We call a *real polynomial $P(x)$ a (normalized) g-extremal polynomial if all its critical points, except g of them, are simple and correspond to the values* ±1. The parameter g involved in this definition (the number of exceptional points) can be calculated by the following formula:

$$g=\sum_{x\colon\, P(x)\neq\pm1}\operatorname{ord}P'(x)+\sum_{x\colon\, P(x)=\pm1}\left[\frac{1}{2}\operatorname{ord}P'(x)\right], \tag{2}$$

where $\operatorname{ord}P'(x)$ is the order of the zero of the derivative of P at the point $x\in\mathbb{C}$ and $[\,\cdot\,]$ is the integer part of a number. Polynomials with 'extremality parameters' $g=0$ and $g=1$ were discovered one and a half centuries ago and are known as the Chebyshev and the Zolotarëv polynomials, respectively. We present the graphs of several 2-extremal polynomials in Fig. 2. For applications to least deviation problems more important are polynomials with small g; they can be parametrized and effectively calculated.

In the present paper we study general g-extremal polynomials. Polynomials (along with rational and algebraic functions) with few critical values are the classical subject of mathematical studies, lying at the juncture between continuous and discrete. One traditional approach in these studies goes back to Hurwitz (1891) and relates to the description of branched covers over a sphere, the investigation of the strata of the resulting discriminant set, of the Lyashko–Loojenga map, and so on. This approach has been actively pursued in recent years; see the comments and the bibliography to problem 1970-15 in 'Arnol'd's problems' [8]. Another tradition goes back to Chebyshev (1853), and in effect to Abel (1826) and relates to the investigation of Pell's equation with polynomial coefficients, to expansions in continued fractions, and to conditions ensuring the degeneracy of hyperelliptic integrals so that they can be expressed in terms of logarithms. A survey of these results can be found in [9]; their characteristic features are effective calculations and connections with applications. Our motivations described in the beginning of this paper explain why we take the second approach and are striving to reduce it to effective numerical calculations [10], [11].

In § 3 we associate with each polynomial $P(x)$ a hyperelliptic curve

$$M=\left\{(x,w)\in\mathbb{C}^2: w^2=\prod_{s=1}^{2g+2}(x-e_s)\right\}$$

ramified over the odd-order zeros of the polynomial $P^2(x)-1$. The genus of the curve M associated with a g-extremal polynomial is equal to the number g of exceptional critical points of the polynomial calculated by formula (2). A polynomial of degree n can be recovered up to a sign from its hyperelliptic curve by the explicit formula

$$P_n(x)=\pm\cos\biggl(ni\int_{(e,0)}^{(x,w)}d\eta_M\biggr), \qquad x\in\mathbb{C}, \quad (x,w)\in M,$$

where $d\eta_M$ is an Abelian differential of the 3rd kind with poles at infinity uniquely assigned to M. The last formula generalizes the classical representations of the Chebyshev and the Zolotarëv polynomials in terms of sines and elliptic functions, respectively. We see that for small g extremal polynomials can be described by means of few parameters, the moduli of the associated curve. Unfortunately, these parameters are not free, but must satisfy certain relations.

The curves M generated by polynomials of degree n satisfy Abel's equations

$$\int_{C_s^-}d\eta_M=2\pi i\frac{m_s}{n}\,, \qquad s=0,1,\dots,g,$$

considered in § 4, in which the C_s^- form a basis in the lattice of odd 1-cycles on the curve M and the m_s are certain integers. It is convenient to assume that the polynomial P is taken to a point in the $2g$-dimensional real space $\mathcal{H}_g$, the *moduli space* of real hyperelliptic curves of genus g with distinguished point on the real oval. The left-hand sides of Abel's equations, the periods of the differential $d\eta_M$ assigned to the curve, extend in the natural way to the components $\mathcal{H}_g^k$, $k=0,\dots,g+1$, of the moduli space. Although the resulting map is multivalued, this complication can be overcome by proceeding to the universal cover $\widetilde{\mathcal{H}}_g^k$ on which the *period map* is a submersion (a map of maximum rank). Polynomials of degree n correspond in the moduli space to smooth g-dimensional fibres of the period map projecting onto the lattice defined by the right-hand side of Abel's equation. As $n\to\infty$, this lattice becomes dense in the range of the map, therefore each neighbourhood of an arbitrary point in the moduli space contains points M corresponding to polynomials (maybe of high degree).

Associating curves with polynomials in this way one constructs a hierarchy whose gth grade consists of polynomials related to curves of genus g and depending on g integer-valued parameters $m_1,\dots,\ m_g$ and on g continuous parameters, local coordinates in the fibre of the period map.

How can one solve concrete least deviation problems in the framework of this approach? A formula for the recovery of a polynomial P_n from its associated curve M can be made effective via Riemann theta-functions of Schottky automorphic functions [10]. Hence one can solve extremal problems by picking an appropriate substitution (=Ansatz). *First*, one must 'guess' the Ansatz, that is, analyse the problem and find the values of the discrete parameters g and $m_0, m_1, \dots, m_g$ corresponding to its solution. This corresponds to determining the low-dimensional face of the ball in the space of polynomials (1) containing the solution P_n. *Next*, one must make up and solve numerically $2g$ transcendental equations for the continuous parameters of the Ansatz, the moduli of the curve M associated with the solution. This system includes Abel's equations and the data of the optimization problem: the constraints, the boundary of the set E. Conceptually and technically, this approach is more complicated than the ones mentioned before. Its advantages are as follows: the complexity of the computation of the solution P_n by explicit analytic formulae does not depend on the degree n of the polynomials, which is clear in the case of Chebyshev's and Zolotarëv's classical formulae. On the other hand, the amount of calculations rapidly increases with g, therefore the natural domain of this method is solutions of large degree n with few constraints for their coefficients and the set E consisting of few components.

§ 2. Least deviation problems

The most common solutions of least deviation problems are polynomials with normalizations that are extremal in the sense of the above definitions. The reason for this is explained by convex analysis [1], [7]. We now discuss Problem A from the introduction.

Assume that we look for the solution of Problem A in the space (1) on a fixed affine plane L_{n+1-r} of codimension r. Such an $(n+1-r)$-plane can be described as a translation of the annihilator of an r-dimensional subspace L^*_r of the dual space. With each non-trivial polynomial $T(x)$ in (1) we associate a convex polyhedral cone in the same dual space, which we defined below. By the extremal points of the polynomial $T(x)$ with respect to E we shall mean the set

$$\mathrm{ext}_E(T) := \{x \in E : T(x) = \pm\|T\|_E\},$$

where we associate with each point x a functional x^* on the space of polynomials: $\langle x^*|P\rangle := P(x)\cdot \operatorname{sign} T(x)$. The *conical hull* of these functionals

$$\mathrm{cone}\{x^*_1, x^*_2, \dots, x^*_m\} := \biggl\{\sum_{s=1}^{m} \alpha_s x^*_s : \alpha_s \geqslant 0,\ \sum_{s=1}^{m} \alpha_s > 0\biggr\},\ \ m = \#\,\mathrm{ext}_E(T), \quad (3)$$

does not contain the origin because $\bigl\langle \sum_{s=1}^m \alpha_s x^*_s \mid T\bigr\rangle = \sum_{s=1}^m \alpha_s \|T\|_E > 0$; we associate it with the polynomial T. By the *dual cone* we shall mean the cone of polynomials positive at each functional in (3) (note that only the non-negativity is required in the standard definition).

Theorem 1. *A polynomial $T \in L_{n+1-r}$ delivers a minimum in the least deviation Problem* A *if and only if the director plane L^*_r intersects the cone* (3) *associated with the polynomial.*

Remark. The cone in this statement, which is generated by all extremal points of the polynomial T can, in view of Carathéodory's principle, be replaced by a cone generated by at most $n+2-r$ extremal points. Thus adjusted, Theorem 1 becomes an interpretation of the criterion of extremality in [1].

Proof. A polynomial $T \in L_{n+1-r}$ fails to be a solution of Problem A if and only if the norms of the polynomials decrease along some ray issuing from T in a direction tangent to the plane L_{n+1-r}. Such a direction can be defined by a polynomial $P(x)$ annihilating all functionals in L_r^* and taking values of the same sign as T at the extremal points of T. The following two assertions are therefore equivalent: T is a solution of Problem A and the annihilator of L_r^* is disjoint from the cone dual to (3). Dualizing the second assertion one obtains the result of the theorem.

(1). Assume that the intersection of the lineal L_r^* with the cone (3) is non-empty and contains a functional p^*. Taking the intersection of the dual objects, $(L_r^*)^\perp \cap \{\text{the dual cone}\} \ni P(x)$ leads to a contradiction: $0 = \langle p^* \mid P \rangle > 0$.

(2) Assume that L_r^* is disjoint from the cone (3); recall that the latter does not contain its vertex. Using induction we now increase L_r^* to a hypersurface disjoint from the cone. It is the annihilator of a polynomial belonging to both the dual cone and the annihilator of L_r^*. At each step, if $r < n$, then we consider a two-dimensional subspace L_2^* linearly independent of L_r^*. Its intersection with the convex cone $L_r^* + \operatorname{cone}\{x_1^*, x_2^*, \ldots, x_m^*\}$ is a convex two-dimensional sector with opening less than π for it does not contain the origin. Hence L_2^* contains a one-dimensional subspace L_1^* disjoint from $L_r^* + \operatorname{cone}\{x_1^*, x_2^*, \ldots, x_m^*\}$. Setting $L_{r+1}^* := L_1^* + L_r^*$ we complete the induction step.

Problems A on least deviation for a fixed subset E of the real axis differ in the position of the plane L_{n+1-r} and therefore can be indexed by points in the real projective Grassmannian $\operatorname{Gr}(n+2, n+2-r)$ of dimension $r(n+2-r)$. We see that more problems can be posed than there exist solutions, so that the question of the rate of the occurrence of each polynomial in (1) among the solutions of least deviation problems suggests itself. The affine planes L_{n+1-r} incident to a fixed point in the space will be indexed by the directing lineals $L_r^* \in \operatorname{Gr}(n+1, r)$.

Theorem 2. *A fixed polynomial T is a solution of a set of Problems* A *corresponding in the Grassmannian* $\operatorname{Gr}(n+1, r)$ *to a closed subset with non-empty interior of a Schubert cycle of codimension* $\max(n+2-r-\#\operatorname{ext}_E(T), 0)$.

Proof. The lineals L_r^* intersecting the cone (3) form a closed subset of the Grassmannian $\operatorname{Gr}(n+1, r)$, because this cone becomes closed after the addition of its vertex. The set in question lies in some Schubert cycle.

Assume that the linear span of the cone (3) belongs to a filtration of the dual space of (1):

$$0 \subset \mathbb{R}^1 \subset \mathbb{R}^2 \subset \cdots \subset \mathbb{R}^{n+1}.$$

If a subspace L_r^* intersects the cone, then we obtain the first inequality in the following system (while the other inequalities hold by dimensional considerations):

$$\dim(L_r^* \cap \mathbb{R}^{\min(m,n+1)}) \geqslant 1, \qquad \dim(L_r^* \cap \mathbb{R}^{n+s+1-r}) \geqslant s, \quad s = 1, 2, \ldots, r,$$

which means that L_r^* lies in the Schubert cycle whose Young diagram is a $(n+1-r) \times r$ rectangle without the (horizontal) row of length $\max(n+2-r-m, 0)$

in the lower right corner. We shall now indicate a subdomain of this Schubert cycle the elements of which intersect the cone (3).

In the proof of Theorem 1 we established the existence of a support hyperplane of the cone (3) containing its origin, but disjoint from the cone proper. Consider now an arbitrary subspace L_l^* of dimension $l := \min(m, n+2-r) - 1$ lying in the intersection of this hyperplane and the linear span of the cone. For a fixed point p^* in the relative interior of the cone there exists a neighbourhood of the origin $\mathcal{O} \subset L_l^*$ such that $p^* + \mathcal{O}$ lies in the cone. We consider now the set of pairs (y^*, L_{r-1}^*), where $y^* \in \mathcal{O}$ and L_{r-1}^* is a subspace of the support hyperplane such that $\dim(L_{r-1}^* \cap L_l^*) = 0$. Such subspaces L_{r-1}^* fill an open subset of the Grassmannian $\mathrm{Gr}(n, r-1)$, which contains at any rate a Schubert cell of the maximum dimension $(r-1)\cdot(n+1-r)$. Each pair (y^*, L_{r-1}^*) defines an r-subspace spanned by L_{r-1}^* and the vector $p^* + y^*$ and intersecting the cone. By construction distinct pairs define distinct r-subspaces. We have thus defined an embedding in the set of r-subspaces intersecting the cone (3) of a domain in the space of dimension

$$(r-1)\cdot(n+1-r) + l = r(n+1-r) - \max(n+2-r-m, 0)$$

equal to the dimension of the Schubert cell in the previous paragraph.

Of course, our arguments do not mean that a slight perturbation of the conditions of an arbitrary problem brings the number of extremal points of the solution T (which may also be non-unique) close to the expected quantity $n+2-r$. Although each polynomial with $\#\,\mathrm{ext}_E(T) < n+2-r$ is a solution of fewer problems, the number of such polynomials is much greater. A crude dimension evaluation shows that these two effects roughly counterbalance each other: in (1) the polynomials with $\#\,\mathrm{ext}_E(T) = m$ lie on submanifolds of codimension $m-1$ and each is a solution of an $((r-1)\cdot(n+1-r)+m-1)$-dimensional set of problems, which yields precisely the dimension of the Grassmannian $\mathrm{Gr}(n+2, n+2-r)$, the index set of least deviation problems.

Which least deviation problems automatically have extremal polynomials as solutions? Each extremal point of a polynomial T lying in the interior of E is critical, and the value $\pm\|T\|_E$ at this point has even multiplicity. That is, we are interested in problems whose solutions have many extremal points, provided that the boundary of E consists of few points. For instance, the number of extremal points of a solution is at least $n+2-r$ if the polynomials satisfying the homogeneous constraints of the problem form a *Chebyshev* subspace. This means that a non-trivial polynomial in $(L_r^*)^\perp$ has at most $\dim(L_r^*)^\perp - 1 = n-r$ zeros in E. The main source of Chebyshev subspaces is provided by *divisor spaces* occurring in algebraic geometry. Let D be a divisor (that is, a formal finite sum of points with integer multiplicities) in the Riemann sphere that is symmetric relative to the real axis and assume that $\mathsf{D} + n\cdot\infty \geqslant 0$. By the space of this divisor we shall mean the subspace of polynomials in (1) such that the multiplicities of their zeros at an arbitrary point in the Riemann sphere (a pole has a negative multiplicity) are no smaller than the multiplicity of this point in the divisor:

$$\mathcal{L}(-\mathsf{D}) := \{P \in \mathbb{R}[x] : (P) \geqslant \mathsf{D}\}, \qquad \operatorname{codim} \mathcal{L}(-\mathsf{D}) = \min(\deg \mathsf{D} + n, n+1). \quad (4)$$

If the support of the divisor D is disjoint from the set E, then the divisor space is Chebyshev on E. The constraints in the corresponding least deviation problem fix the values of the solution T at the finite points in D (and the values of its first derivatives if the multiplicity of the point is larger than 1), and also fix several leading coefficients of T if the point at infinity has multiplicity greater than $-n$ in the divisor.

Theorem 3 [2]. (1) *If a lineal $(L_r^*)^\perp$ is Chebyshev on a set E, then the solution of the corresponding least deviation problem* A *has at least $n+2-r$ extremal points in E.*

(2) *If the same lineal is Chebyshev on the convex hull of E, then the solution is unique and is characterized by the property of having an $(n+2-r)$-alternance on E.*

Proof. (1) If $T(x)$ is a solution of the least deviation problem, then by Theorem 1 for some extremal point x_s of this polynomial and *positive* weights α_s we obtain

$$\sum_{s=1}^{m} \alpha_s \cdot \operatorname{sign} T(x_s) \cdot P(x_s) = 0 \quad \text{for each } P \in (L_r^*)^\perp. \tag{5}$$

Assume that there exist $m < n+2-r$ extremal points. The dimension of $(L_r^*)^\perp$ is $n+1-r$, therefore there exists a polynomial $P(x) \in (L_r^*)^\perp$ vanishing at $n-r$ points: at $x_1, x_2, \dots, x_{m-1}$ and at arbitrary $n+1-r-m$ points in $E \setminus \operatorname{ext}_E(T)$. Since $(L_r^*)^\perp$ is a Chebyshev space, it follows that $P(x_m) \neq 0$, therefore in (5) we have $\alpha_m = 0$, a contradiction.

(2) Assume that the solution $T(x)$ has the same sign at two neighbouring points x_s and x_{s+1}. We consider a polynomial $P(x) \in (L_r^*)^\perp$, vanishing at the remaining $n-r$ extremal points (see the remark to Theorem 1). Then equality (5) takes the form $\alpha_s P(x_s) + \alpha_{s+1} P(x_{s+1}) = 0$. This means that $P(x)$ must also have a zero on the interval $[x_s, x_{s+1}]$, in contradiction with the Chebyshev property of the space $(L_r^*)^\perp$ on $\operatorname{conv} E$. Thus, each solution T has an $(n+2-r)$-alternance on E. Conversely, each polynomial $T(x) \in L_{n+1-r}$ having an $(n+2-r)$-alternance on E is a unique solution. If there exists another polynomial whose deviation on E is not worse than that of $T(x)$, then their difference belongs to $(L_r^*)^\perp$ and has at most $n+1-r$ zeros on $\operatorname{conv} E$, so that it is trivial.

The Lebedev problem is not formally a least deviation problem, but its solution is at the same time a solution of a certain Problem A, therefore convex analysis enables one to determine the form of the solution of Problem B.

Theorem 4. *For $r > 1$ Problem* B *is uniquely soluble and its solution has an $(n+2-r)$-alternance on $E \setminus \{0\}$.*

Proof. As t increases, the unit ball in (1) with respect to the norm of $C[0,t]$ (linearly, but anisotropically) contracts and, in the limit as $t \to \infty$, contains only constant polynomials, which cannot satisfy the constraints if $r > 1$:

$$P_n(x) = 1 - x + \frac{x^2}{2} - \dots + \frac{(-x)^{r-1}}{(r-1)!} + x^r P_{n-r}(x), \quad \deg P_{n-r}(x) \leqslant n-r. \tag{6}$$

Hence there exist a largest interval $E := [0, t^0]$ and a polynomial $T(x)$ with deviation 1 on E satisfying the constraints. We claim that $T(x)$ is at the same time a solution of Problem A with constraints (6) on the interval $E' = [\varepsilon, t^0]$; here ε is a positive quantity that is smaller than 1, $t^0/2$, and $1/\max|P''(x)|$, where the maximum is considered over the compact set

$$\{(P, x) : x \in [0, t^0/2],\ \|P\|_{[t^0/2, t^0]} \leqslant 1,\ \deg P \leqslant n\}.$$

Assume that there exists a polynomial $P(x)$ with behaviour (6) and deviation less than 1 on E'. In view of the local decrease of $P(x)$ in the neighbourhood of the origin and the smallness of ε, $\|P\|_E \leqslant 1$. Since the value of $P(x)$ at the end-point $x = t^0$ is less than 1 in absolute value, E can be increased while keeping the norm of $P(x)$ the same, which contradicts the maximality of E.

The linear subspace corresponding to the constraints (6) is defined by a divisor D with support at the origin and at infinity, which does not intersect E'. By Theorem 3(2) the solution $T(x)$ is unique and has an $(n + 2 - r)$-alternance on E'.

§ 3. Chebyshev representation of polynomials

Chebyshev and his students Zolotarëv, the brothers V. A. and A. A. Markov, Korkin, and Posse reduced extremal problems for polynomials to Pell's equation, a geometric interpretation of which is suggested in the following construction.

Construction. One associates with an arbitrary polynomial $P(x)$ the two-sheeted Riemannian surface

$$M = M(\mathbf{e}) = \left\{(x, w) \in \mathbb{C}^2 : w^2 = \prod_{s=1}^{2g+2} (x - e_s)\right\} \tag{7}$$

ramified over the points $\mathbf{e} := \{e_s\}_{s=1}^{2g+2}$ at which the polynomial takes the values ± 1 with odd multiplicity (that is, these are simple values in the general case).

Discussion. The motivation of the construction [10], [12] is purely topological. Consider the diagram

$$\begin{array}{ccc} (x, w) \in M(\mathbf{e}) & \xrightarrow{\widetilde{P}} & \mathbb{CP}_1 \ni u \\ \chi\big\downarrow & & \big\downarrow\sigma \\ x \in \mathbb{CP}_1 & \xrightarrow{P} & \mathbb{CP}_1 \end{array} \tag{8}$$

in which: $\chi(x, w) := x$ is a two-sheeted cover ramified over the points in $\mathbf{e}$; $\sigma(u) := \frac{1}{2}\bigl(u + \frac{1}{u}\bigr)$ is a two-sheeted cover ramified over ± 1 (the Zhukovskiĭ function). Each map $\widetilde{P}$ between the covering spaces satisfying the equivariance condition $\widetilde{P}(x, -w) = 1/\widetilde{P}(x, w)$ can be lowered to a map P of the bases which has an odd branching index at the points in $\mathbf{e} \subset P^{-1}(\pm 1)$ and an even index at the points in $P^{-1}(\pm 1) \setminus \mathbf{e}$. The converse result also holds: each map P ramified with an even order over ± 1, except at the points in $\mathbf{e}$, can be lifted to an equivariant map of the covering spaces in the diagram (8). In this way we obtain a description of extremal polynomials P using few parameters, the modules of the curve M. Almost all the critical points of the polynomial P generated by an equivariant map of the covering spaces are automatically simple and correspond to the values ± 1.

Lemma 1. *The genus of the curve associated with a polynomial $P(x)$ is equal to the integer g in* (2).

Proof. A polynomial P of degree n has $n-1$ critical points in $\mathbb{C}$:

$$n-1=\sum_x \operatorname{ord} P'(x) = g + \sum_{x:P(x)=\pm 1} \left[\frac{1}{2}(\operatorname{ord} P'(x)+1)\right].$$

We now calculate $\deg \mathbf{e}$, the number of odd-order zeros of the polynomial $P^2(x)-1$:

$$2n=\sum_{x:P(x)=\pm 1} (\operatorname{ord} P'(x)+1) = \deg \mathbf{e} + \sum_{x:P(x)=\pm 1} 2\left[\frac{1}{2}(\operatorname{ord} P'(x)+1)\right],$$

which shows that $\deg \mathbf{e} = 2g(M)+2$, that is, the genus of the hyperelliptic curve $M(\mathbf{e})$ is g.

Example. Let E be a closed interval and assume that the r constraints in extremal Problem A define a Chebyshev subspace $(L_r^*)^\perp$. Then the normalized solution $P_n(x) := T_n(x)/\|T_n\|_E$ corresponds to a curve M of genus $g \leqslant r-1$. In fact, the full inverse image $P_n^{-1}(\pm 1)$ contains $2n$ points counted with multiplicities. At least $n-r$ points of even multiplicity from the inverse image lie in the interior of E. At the $2g+2$ ramification points of M the value of P_n has odd multiplicity, therefore $2n \geqslant 2(n-r)+2g+2$.

3.1. Real hyperelliptic curves. We recall several concepts of the geometry of hyperelliptic curves [13]. A compact complex curve M_c of genus g is said to be *hyperelliptic* if it admits a *conformal involution* J with $2g+2$ fixed points. If $g>1$, then such an involution is unique (if it exists), while for $g=0,1$ there exist infinitely many involutions J. A curve M_c is said to be *real* if it admits an *anticonformal involution* $\bar{J}$ (a reflection). Whatever the genus, there can exist several anticonformal involutions, therefore one must consider a pair $(M_c,\bar{J})$. We now discuss the connections between these concepts. If a curve M_c admits a hyperelliptic involution J and an anticonformal involution $\bar{J}$, then they commute for $g>1$ ($\bar{J}J\bar{J}$ is another hyperelliptic involution). This is in general not so for $g=0,1$, but we shall assume that $\bar{J}J = J\bar{J}$. The interchangeability of the involutions means that $\bar{J}$ acts on $\mathbb{CP}_1 = M_c/J$. The anticonformal involution of the Riemann sphere interchanges the interior and the exterior of its *isometric circle*. The points on this circle are either fixed (for instance, when $\bar{J}x = 1/\bar{x}$) or are taken to the antipodal points (for instance, when $\bar{J}x=-1/\bar{x}$). Hence the real hyperelliptic curves fall into two classes: with orientable quotient $M_c/\langle J,\bar{J}\rangle$ (= a disc) and with non-orientable one (= the projective plane). In what follows we consider only the first class, of *real orientable* hyperelliptic curves.

We lift the circle of $\bar{J}$-fixed points from the sphere to the curve M_c. On the curve we obtain *real ovals* that are fixed under the $\bar{J}$-action on M_c and *coreal ovals* that are fixed under the action of $\bar{J}J$. If there exist $2k$ ramification points (fixed points of the involution J), $k=0,1,\dots,g+1$, projecting onto the circle of fixed points in the sphere, then for $k>0$ there exist on M_c precisely k real and k coreal ovals with alternating projections onto the circle of fixed points.

The case $k = 0$ drops out of the general picture: for real g there exists only one oval, either real or coreal, while for odd g there exist two ovals of the same name.

Real orientable hyperelliptic curves have a convenient algebraic model (7) in which all *branch points* e_s are distinct and form the *branch divisor* $\mathbf{e} := \{e_s\}_{s=1}^{2g+2}$ symmetric with respect to the real axis. In Fig. 1 we plot by bold lines the *system of cuts* Λ on $\mathbb{C} \setminus \mathbf{e}$ outside which the function $w(x)$ has a single-valued branch. The curve $M(\mathbf{e})$ can be thought of as two sheets of $\mathbb{C} \setminus \Lambda$ glued crosswise along the cuts. The compactification M_c of the curve (7) is obtained by the addition of a pair of points $\infty_\pm$ at the infinity of each sheet. In this model the hyperelliptic and the anticonformal involutions have the following form: $J(x, w) := (x, -w)$, $\bar{J}(x, w) := (\bar{x}, \bar{w})$. For such a choice of $\bar{J}$ the punctures $\infty_\pm$ lie on a real oval and the topological invariant k of the real curve M can be defined as the number of coreal ovals on it.

3.1.1. Homology space and the lattice L_M. The curve M can be obtained from the compact curve by the deletion of the two points at infinity, therefore one must add to the usual $2g$ independent 1-cycles the cycle encircling an (arbitrary) puncture. In the $(2g+1)$-dimensional real *homology space* $H_1(M, \mathbb{R})$ we obtain an action of the anticonformal involution $\bar{J}$, which decomposes it into the sum of the eigenspaces $H_1^\pm(M, \mathbb{R})$ corresponding to the eigenvalues ± 1. The *even* 1-cycles C satisfy the equality $\bar{J}C = C$ and form the g-dimensional subspace $H_1^+(M)$. For $k > 0$ one can take for the first $k-1$ cycles of a basis $C_1^+, C_2^+, \dots, C_g^+$ in this space the finite real ovals, similarly to Fig. (1)(a) (the dashed line indicates that the contour traverses the lower sheet). The *odd* 1-cycles C, defined by the condition $\bar{J}C = -C$, give rise to the $(g+1)$-dimensional subspace $H_1^-(M)$. The cycles $C_0^-, C_1^-, C_2^-, \dots, C_g^-$ in Fig (1)(b), the first k of which are coreal ovals, form a basis in $H_1^-(M)$. The sum $C_\infty := C_0^- + C_1^- + C_2^- + \dots + C_g^-$ is homologous to the cycle encircling the puncture on the curve at infinity and spans a distinguished 1-dimensional subspace $H_1^\infty(M)$ of $H_1^-(M)$. The restriction of the *intersection form* to the subspaces $H_1^\pm(M)$ is trivial: the involution $\bar{J}$ reverses the orientation, therefore $\bar{J}C \circ \bar{J}C' = -C \circ C'$, $C, C' \in H_1(M, \mathbb{R})$. Only the following entries of the intersection matrix with respect to the above basis are distinct from zero:

$$\begin{aligned} &C_s^+ \circ C_s^- = 1, && s = 1, \dots, k-1, \\ &C_s^+ \circ C_s^- = 2, && s = k, \dots, g, \\ &C_s^+ \circ C_0^- = -C_s^+ \circ C_s^-, && s = 1, \dots, g. \end{aligned} \tag{9}$$

Using the intersection form one can show that the above-considered systems of cycles form bases not only in the Euclidean spaces $H_1^\pm(M, \mathbb{R})$, but also in the *lattices* $H_1^\pm(M, \mathbb{Z}) := H_1^\pm(M, \mathbb{R}) \cap H_1(M, \mathbb{Z})$. Important for the investigation of the Chebyshev construction is the sublattice L_M of the lattice of odd cycles generated by the elements $2C_0^-, 2C_1^-, \dots, 2C_{k-1}^-; C_k^-, C_{k+1}^-, \dots, C_g^-$. For $k = 0$ the lattices L_M and $H_1^-(M, \mathbb{Z})$ are equal, while for $k > 0$ the lattice L_M has the following 'coordinateless' description.

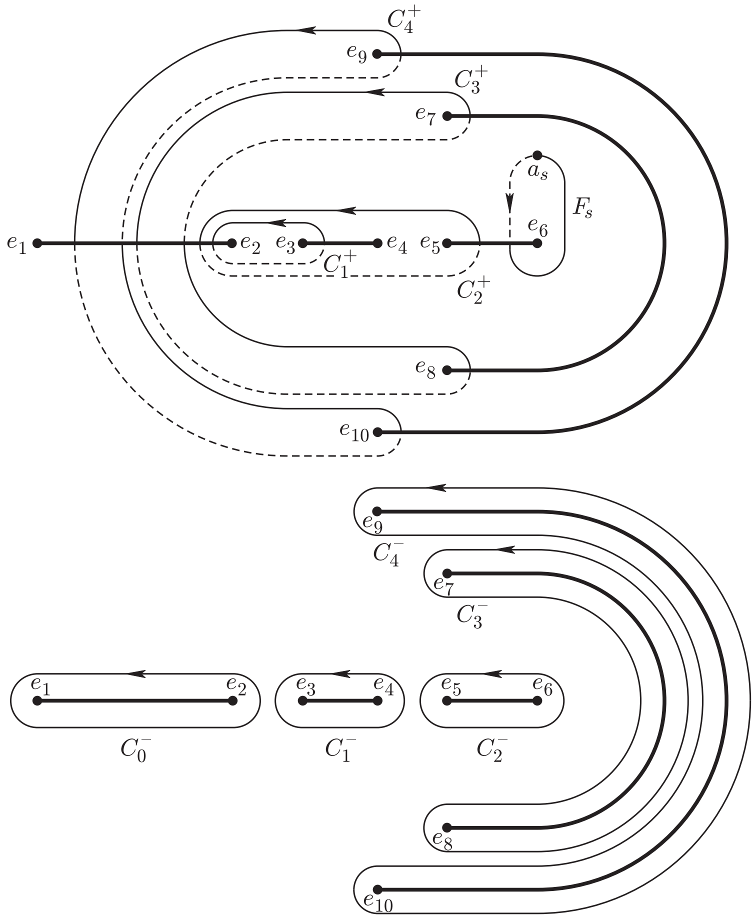

Figure 1. The system of cuts Λ in the plane and the basis in the lattice (a) $H_1^+(M,\mathbb{Z})$, (b) $H_1^-(M,\mathbb{Z})$

Lemma 2. *If $k>0$, then the following two lattices coincide with L_M:*

(1) *the projection of the lattice $2H_1(M,\mathbb{Z})$ onto the subspace $H_1^-(M,\mathbb{R})$ along $H_1^+(M,\mathbb{R})$;*

(2) *the cycles $H_1^-(M,\mathbb{Z})$ having even intersection indices with all components of the real ovals M (the punctures at infinity partition one real oval).*

Proof. The projection of the space $H_1(M,\mathbb{R})$ onto H_1^- along H_1^+ has the form $C\to\frac{1}{2}(C-\bar{J}C)$. The assertions in question can be verified on the generators of the lattices.

3.1.2. Space of differentials on the curve. The *Abelian differentials* $d\xi$ on M the only possible singularities of which are simple poles at infinity $\infty_\pm$ make up a complex linear space of dimension $g+1$. The Riemann bilinear relations [14], in view of what is known about the intersection form of the homology basis, ensure that there exists in this space a unique meromorphic differential with fixed residue at infinity and fixed g periods over the cycles in the basis $H_1^+(M)$ (or in the basis $H_1^-(M)/H_1^\infty(M)$). The anticonformal involution $\bar{J}$ of the curve also acts in the space of differentials: $d\xi \to \bar{J}\,\overline{d\xi}$. Fixed points of this action make up what is usually called [13] the space of *real differentials*. In the model (7) these differentials have the representation $d\xi = w^{-1}P_g(x)\,dx$ with real polynomial $P_g(x)$ of degree at most g; they take real values on the cycles in $H_1^+(M)$ and purely imaginary values on the cycles in $H_1^-(M)$. The *period map* $\Pi(d\xi)$:

$$\langle \Pi(d\xi) \mid C^+ + C^- \rangle := \int_{C^+} d\xi - i\int_{C^-} d\xi, \qquad C^\pm \in H_1^\pm(M,\mathbb{R}), \tag{10}$$

assigns to each real differential $d\xi$ an element of the real cohomology group $H^1(M,\mathbb{R})$ of the curve M.

The space under consideration contains a unique differential $d\eta = d\eta_M$ with residue -1 at the point ∞_+ in the upper sheet of M and with period zero over all real 1-cycles. We can verify that this differential is real: the meromorphic differential $\bar{J}\,\overline{d\eta_M}$ has residues ∓ 1 at the points $\infty_\pm$ and periods zero over even cycles. By the uniqueness of the differential so normalized we obtain $\bar{J}\,\overline{d\eta_M} = d\eta_M$. The differential $d\eta_M$ associated with the curve can be also characterized by the property that all its periods on M are purely imaginary.

3.2. Polynomials and curves. The following result describes the range of the Chebyshev map taking polynomials to curves and the inverse map.

Theorem 5. *The construction described in the beginning of § 3 establishes a one-to-one correspondence between real polynomials $P_n(x)$ of degree n considered up to the sign and real orientable hyperelliptic curves M for which the period map of the differential $d\eta_M$ associated with the curve yields a $4\pi n^{-1}\mathbb{Z}$-valued functional on the lattice L_M. The polynomial can be recovered from the associated curve M by the formula*

$$P_n(x) = P_n(e_s)\cos\left(ni\int_{(e_s,0)}^{(x,w)} d\eta_M\right), \tag{11}$$

where the result is independent of the integration path on M, of the choice between the two possible values of $w(x)$, and of the branch point e_s, $s = 1,\dots,2g+2$, taken for the initial point of integration.

Remark. For $k > 0$, in the case important for applications to extremal polynomials, there exists an equivalent and more easily understandable condition describing the curves M corresponding to polynomials of degree n: $\Pi(d\eta_M) \in 2\pi n^{-1}H^1(M,\mathbb{Z})$.

Proof. (1) *The correspondence $P_n \to M$.* If a curve M of the form (7) corresponds to a polynomial $P_n(x)$, then there exists a real polynomial $P_{n-g-1}(x)$ such that $P_n^2(x) - 1 = w^2P_{n-g-1}^2(x)$. On the compact curve M_c we consider the function $\widetilde{P}(x,w) := P_n(x) + wP_{n-g-1}(x)$ introduced by N. Akhiezer and satisfying the condition of equivariance with respect to the covering groups of the covers χ and σ in the diagram (8): $\widetilde{P}(x,-w) = P_n(x) - wP_{n-g-1}(x) = 1/\widetilde{P}(x,w)$. The meromorphic differential

$$d\eta := \frac{1}{n}\,\frac{d\widetilde{P}}{\widetilde{P}}$$

is equal to the differential $d\eta_M$ associated with the curve. In fact, the only singularities of $d\eta$ are simple poles at infinity, with residues ± 1. All the periods of $d\eta$ are purely imaginary because on the closed contours $C \subset M$ we have

$$\int_C d\eta = n^{-1} \log \widetilde{P}(x,w)\big|_C \in \frac{2\pi i}{n}\mathbb{Z}.$$

We claim that on the lattice L_M the functional $\Pi(d\eta_M)$ takes values in $4\pi\mathbb{Z}/n$. We now calculate the index of the intersection of the image $\widetilde{P}(C)$ of an arbitrary contour C in this lattice with the positive half-axis. Slightly modifying the contour in its homology class we can assume that all its intersections with the set $\{\widetilde{P}(x,w) > 0\} = \{P_n(x) \geqslant 1\}$ are transversal. The latter set is $\bar{J}$-invariant, therefore outside real ovals it intersects the cycle $C = -\bar{J}C$ at an even number of points. On each component of real ovals $P_n^2 \geqslant 1$, and the intersection index of the cycle with these components is also even. This demonstrates that the increment of $\log \widetilde{P}$ over a cycle C in the lattice L_M is an even multiple of $2\pi i$.

Inversion formula (11) follows from the equalities

$$\begin{aligned} P_n(x) &= \frac{1}{2}\big(\widetilde{P}(x,w) + \widetilde{P}(x,-w)\big) = \cos\big(i \log \widetilde{P}(x,w)\big) \\ &= \cos\biggl(i \log \widetilde{P}(e_s,0) + ni \int_{(e_s,0)}^{(x,w)} d\eta_M\biggr) = P_n(e_s)\cos\biggl(ni \int_{(e_s,0)}^{(x,w)} d\eta_M\biggr). \end{aligned}$$

(2) *The correspondence* $M \to P_n$. If the curve M satisfies the assumptions of the theorem, then the functional $\Pi(d\eta_M)$ is $2\pi\mathbb{Z}/n$-valued on the (integer) cycles. For if $C \in H_1(M,\mathbb{Z})$, then the cycle $C - \bar{J}C$ belongs to the lattice L_M, therefore $\langle \Pi(d\eta_M) \mid C\rangle = \frac{1}{2}\langle \Pi(d\eta_M) \mid C - \bar{J}C\rangle \in 2\pi\mathbb{Z}/n$.

For $P_n(e_s) = \pm 1$ the right-hand side of (11) well-defines a meromorphic function on M_c that is stable under involution and has poles of order n at infinity. This is a polynomial of degree n in x, and it is real because

$$\begin{aligned} P_n(\bar{x}) &= P_n(e_s) \cos ni \biggl(\int_{(e_s,0)}^{(\bar{e}_s,0)} d\eta_M + \int_{(\bar{e}_s,0)}^{(\bar{x},\bar{w})} d\eta_M \biggr) \\ &\overset{(*)}{=} P_n(e_s) \cos ni \overline{\int_{(e_s,0)}^{(x,w)} d\eta_M} = \overline{P_n(x)}; \end{aligned}$$

in the transition $(*)$ we use the inclusion $\displaystyle\int_{(e_s,0)}^{(\bar{e}_s,0)} d\eta_M \in 2\pi i\mathbb{Z}/n$, and the fact that $d\eta_M$ is a real differential. It is easy to verify that $P_n(x)$ takes values ± 1 of odd multiplicity at the branch points of the curve M and only at these points.

§ 4. Abel's equations

We now study the structure of the set of curves M associated with polynomials of degree n by means of the Chebyshev correspondence. In view of Theorem 5, the branch points of curves of this type having a fixed genus g are constrained by the following relations:

$$\int_{C_s^-} d\eta_M = 2\pi i \frac{m_s}{n}, \qquad m_s \in \begin{cases} \mathbb{Z}, & s = 0, 1, \dots, k-1, \\ 2\mathbb{Z}, & s = k, \dots, g, \end{cases} \tag{12}$$

where $\{C_s^-\}_{s=0}^g$ is the basis in the lattice $H_1^-(M, \mathbb{Z})$ fixed above. The Riemann bilinear relations, in view of the fact that $d\eta_M$ has periods zero over the cycles in $H_1^+(M)$, bring the system of equations (12) to Abel's classical criterion for the existence of a meromorphic function[1] on M with divisor $n \cdot (\infty_+ - \infty_-)$. Only g of Abel's equations (12) are independent because the cycle $\sum_{s=0}^g C_s^-$ contracts to a pole of $d\eta_M$ with a fixed residue. The left-hand sides of Abel's equations are locally single-valued analytic functions of the branch points of the curve; however, they are globally multivalued: interchanging two branch points in the upper half-plane one obtains another basis in the lattice of odd cycles. This laxity is described in terms of braids acting on the universal cover of the space of curves. We now introduce the requisite concepts.

4.1. Moduli spaces of curves. We fix the topological invariants of a real orientable hyperelliptic curve: its genus $g \geqslant 0$ and the number k of coreal ovals $0 \leqslant k \leqslant g+1$. Let $\mathbf{e}$ be an unordered system of distinct points $e_1, \dots, e_{2g+2}$ including $2k$ real points and $g-k+1$ pairs of complex conjugate points. We have a free action of the group $\mathfrak{A}_1^+$ of orientation-preserving affine motions of the real axis on such systems: $\mathbf{e} = \{e_s\}_{s=1}^{2g+2} \to A\mathbf{e} + B = \{Ae_s + B\}_{s=1}^{2g+2}$, $A > 0$, $B \in \mathbb{R}$. We call the orbits of this action the space $\mathcal{H}_g^k$. Associating each symmetric simple divisor $\mathbf{e}$ with a hyperelliptic curve (7) we arrive at the equivalent definition of $\mathcal{H}_g^k$ as the *space of moduli (= conformal classes) of real orientable hyperelliptic curves of genus g with k coreal ovals and with distinguished point ∞_+ on an oriented real oval that is distinct from the ramification points.*

The space $\mathcal{H}_g^k$ is a $2g$-dimensional real manifold. For the introduction of coordinates we locally number the points in the system $\mathbf{e}$ and fix a pair of complex conjugate points or a pair of real points e_{2g+1}, e_{2g+2}. For local coordinate variables in $\mathcal{H}_g^k$ we take the variables $\operatorname{Re} e_s$ in the case of real points e_s and $\operatorname{Re} e_s$, $\operatorname{Im} e_s$ in the case of points e_s in the real half-plane $\mathbb{H}$, $1 \leqslant s \leqslant 2g$.

Lemma 3. *The fundamental group of the moduli space $\pi_1(\mathcal{H}_g^k)$ is isomorphic to the $(g-k+1)$-string braid group of the plane $\mathrm{Br}_{g-k+1}(\mathbb{H})$.*

Proof. Elements of $\mathcal{H}_g^k$ are orbits of the affine group $\mathfrak{A}_1^+$. For $k > 0$ each orbit contains a unique divisor $\mathbf{e}$ such that the extreme points of the set $\mathbb{R} \cap \mathbf{e}$ are ± 1. For $k < g+1$ the orbit contains a unique divisor $\mathbf{e}$ such that the barycentre of the set $\mathbb{H} \cap \mathbf{e}$ is at i. Thus, the moduli space $\mathcal{H}_g^k$ can always be embedded in the space of symmetric divisors $\mathbf{e}$ with the same invariants g and k. This space of divisors can be contracted to $\mathcal{H}_g^k$, therefore they have equal fundamental groups. A symmetric divisor is completely defined by its parts lying in $\mathbb{H}$ ($g-k+1$ points) and $\mathbb{R}$ ($2k$ points), therefore the space of such divisors is the product of the quotient of the space $\mathbb{H}^{g-k+1} \setminus \{\text{the diagonals}\}$ by the rearrangements of the variables and a $2k$-dimensional cell. The fundamental group of such a space [15] is precisely the braid group $\mathrm{Br}_{g-k+1}(\mathbb{H})$.

[1] The Akhiezer function $\widetilde{P}$ of the diagram (8).

The *universal cover* $\widetilde{\mathcal{H}}_g^k$ of the moduli space (= the set of homotopy classes of paths in $\mathcal{H}_g^k$ starting at the distinguished point $M_0 = M(\mathbf{e}^0)$) is topologically a $2g$-cell.

4.2. Bundles. Over the moduli space $\mathcal{H}_g^k$ we consider two vector bundles: the *bundle of homology groups* $H_1\mathcal{H}_g^k$ and the bundle of real meromorphic *differentials* $\Omega^1\mathcal{H}_g^k$. The fibre of the real bundle $H_1\mathcal{H}_g^k$ over a point $M \in \mathcal{H}_g^k$ is the $(2g+1)$-dimensional homology space $H_1(M, \mathbb{R})$ of the curve M. The local trivialization of the bundle of homology groups is described, for instance, in [16]. This bundle splits in the natural way into the sum of subbundles $H_1^+\mathcal{H}_g^k$ and $H_1^-\mathcal{H}_g^k$ the fibres of which are the eigenspaces corresponding to the values ± 1 of the operator of anticonformal involution $\bar{J}$ acting on homology.

It is known that the vector bundle under consideration possesses the natural flat Gauss–Manin connection [16], which allows one to shift homologies to neighbouring fibres. The action of this connection on cycles in $H_1(M, \mathbb{Z})$ can be described as follows. On the two-dimensional model of the curve M we draw a closed contour representing the cycle and lying away from the ramification points. Keeping this contour fixed and perturbing the ramification points we transfer the cycle to nearby curves M. The parallel shift of cycles defined by the Gauss–Manin connection is compatible with the splitting of the homological vector bundle into the subbundles $H_1^{\pm}\mathcal{H}_g^k$ pointed out above and preserves all the integer homology lattices considered before.

The flat connection enables one to define the *action of the braid group* Br_{g-k+1} in the cohomology space $H^1(M_0, \mathbb{R})$ of the distinguished curve. Namely, we define the action of $\beta \in \pi_1(\mathcal{H}_g^k, M_0)$ on a functional $C_* \in (H_1(M_0))^*$ by the formula $\langle \beta \cdot C_* \mid C\rangle := \langle C_* \mid$ the parallel translation of C along a loop in the class $\beta\rangle$, $C \in H_1(M_0)$. The matrix description of the braid group action on functionals in $(H_1^-(M_0))^*$ coincides with the Burau representation [15].

4.2.1. Global period map. The fibre of the second vector bundle $\Omega^1\mathcal{H}_g^k$ over the curve M is the $(g+1)$-dimensional space of *real* meromorphic differentials $d\xi$ on M, the only allowed singularities of which are simple poles at the distinguished points $\infty_\pm$. The period map (10) enables one to couple local sections of the pairs $\Omega^1\mathcal{H}_g^k$ and $H_1\mathcal{H}_g^k$. A transition to the universal cover gives us the global period map described below.

The universal cover $\widetilde{\mathcal{H}}_g^k \to \mathcal{H}_g^k$ applied in the standard fashion [17] to the above-discussed vector bundles produces bundles $H_1\widetilde{\mathcal{H}}_g^k$, $H_1^{\pm}\widetilde{\mathcal{H}}_g^k$, and $\Omega^1\widetilde{\mathcal{H}}_g^k$. The Gauss-Manin connection enables us to identify the fibres of the bundle of homology groups $H_1\widetilde{\mathcal{H}}_g^k$ over the universal covering space with its fibre over the distinguished point $\widetilde{M}_0 \in \widetilde{\mathcal{H}}_g^k$. The bracket in (10) defines now the *global period map* $\Pi\colon \Omega^1\widetilde{\mathcal{H}}_g^k \to H^1(M_0, \mathbb{R})$. The composite of Π and the restriction of functionals to the subspace $H_1^{\bullet}(M_0, \mathbb{R}) \subset H_1(M_0, \mathbb{R})$, where the index $\bullet$ can take the values $+, -, \infty$, defines a *partial* period map $\Pi_\bullet\colon \Omega^1\widetilde{\mathcal{H}}_g^k \to (H_1^{\bullet}(M_0, \mathbb{R}))^*$. The fibres of the period maps are studied in the next subsection.

The simplest period map Π_∞ defines the residue of the differential at infinity. Its typical fibre $\mathcal{N} := \{d\xi : \langle \Pi(d\xi) \mid C_\infty\rangle = 2\pi\}$ is a smooth $3g$-dimensional cell of differentials with residues ± 1 at the distinguished points $\infty_\mp$ in the curve $\widetilde{M} \in \widetilde{\mathcal{H}}_g^k$. The cell $\mathcal{N}$ is partitioned into smooth submanifolds (strata) consisting of the differentials of the following form:

$$d\xi = \prod_{s=1}^{2g+2} (x-e_s)^{\varepsilon_s} \prod_{j=1}^{l} (x-a_j)^{\alpha_j} \frac{dx}{w}, \quad \varepsilon_s \geqslant 0, \ \ \alpha_j \geqslant 1, \ \ \sum_{s=1}^{2g+2} \varepsilon_s + \sum_{j=1}^{l} \alpha_j = g, \tag{13}$$

with fixed orders of zeros, where all the zeros $a_j \neq e_s$ of the differential are distinct and form a symmetric subset relative to the real axis. For local variables on such a $(2g+l)$-dimensional stratum one can take the quantities $\operatorname{Re} e_s$, $\operatorname{Re} a_j$ in the case of real points e_s and a_j and $\operatorname{Re} e_s$, $\operatorname{Re} a_j$, $\operatorname{Im} e_s$, $\operatorname{Im} a_j$ in the case of e_s and a_j lying in the upper half-plane, $s = 1, \dots, 2g$, $j = 1, \dots, l \leqslant g$.

4.3. Properties of the period map. We shall write out the equations of the fibres of the global period map. We fix 1-cycles $C_1^+, C_2^+, \dots, C_g^+$ and $C_1^-, \dots, C_g^-$, C_∞ forming bases in the homology subspaces $H_1^+(M_0)$ and $H_1^-(M_0)$, respectively, where C_∞ encircles the puncture at ∞_+ counterclockwise. Pairing these fibres with differentials by formula (10) gives us $2g+1$ real analytic functions $\gamma_1^+, \gamma_2^+, \dots, \gamma_g^+$ and $\gamma_1^-, \dots, \gamma_g^-$, $\gamma_\infty = -2\pi \operatorname{Res} d\xi\big|_{\infty_+}$ on the bundle $\Omega^1\widetilde{\mathcal{H}}_g^k$. The space $\mathcal{N}$ of differentials with residues ± 1 is defined by the equation $\gamma_\infty = 2\pi$, and the fibres of the partial period maps $\Pi_\bullet$ restricted to $\mathcal{N}$ are defined by the additional equations $\gamma_s^\bullet(d\xi) = \mathrm{const}_s^\bullet$, $s = 1, \dots, g$, $\bullet = +, -$. In view of Theorem 5, we are interested in manifolds along which the functions γ_s^+ vanish and the γ_s^- take fixed values in $2\pi\mathbb{Q}$. We have the following result.

Theorem 6. (1) *The fibres of the maps* $\Pi_\pm \colon \mathcal{N} \to (H_1^\pm(M_0, \mathbb{R}))^*$ *are smoothly embedded* $2g$*-cells projecting without singularities onto the base of the vector bundle* $\Omega^1\widetilde{\mathcal{H}}_g^k$.

(2) *The fibres of* $\Pi_\pm$ *are transversal to the strata* (13) *in the space* $\mathcal{N}$ *and the fibres of* $\Pi_\mp$.

(3) *The 'rational' fibres* Π, *that is, the fibres such that* $\Pi(d\xi) \in 2\pi H^1(M_0, \mathbb{Q})$, *are dense in* $\mathcal{N}$.

Proof. We claim that the functions $\gamma_1^+, \dots, \gamma_g^+$ and $\gamma_1^-, \dots, \gamma_g^-$ defining the fibres of the period map can be complemented on each stratum (13) to a system of local coordinate variables. Hence the differentials of these functions form a $2g$-dimensional subspace of the cotangent space to each $(2g+l)$-dimensional stratum, and therefore they are also linearly independent in the ambient space $\mathcal{N}$. This means that the period map has smooth fibres with transversality property (2) and the 'rational' fibres are dense (3).

We shall now define locally on each stratum (13) other l real analytic functions φ_s, $s = 1, \dots, l$. On the curve M we consider a path F_s connecting two zeros $(a_s, \pm w(a_s))$ of the differential $d\xi$ projecting onto the same point in the x-plane. In a small neighbourhood on the stratum this path can be assumed to continuously depend on $d\xi$: the end-points of F_s 'float' together with the zeros of the differential (see Fig. 1(a)). The choice of F_s is homotopically non-unique, but two possible paths differ by a cycle on M: if the zero a_s lies on the projection of a real oval onto M, then $F_s - \bar{J}F_s \in H_1^-(M,\mathbb{Z})$; if a_s lies in the projection of a coreal oval, then $F_s + \bar{J}F_s \in H_1^+(M,\mathbb{Z})$; for a pair of complex conjugate zeros $a_s, a_{\bar{s}}$ we have $\bar{J}F_s - F_{\bar{s}} \in H_1(M,\mathbb{Z})$.

A fixation of the paths F_s allows one to introduce locally on the stratum l complex-valued functions $f_s := \int_{F_s} d\xi$ whose real and imaginary parts give us the missing coordinate variables. Namely, for each zero a_s in a real oval on M we set $\varphi_s := \operatorname{Re} f_s$; for a_s in the coreal oval on M we set $\varphi_s := \operatorname{Im} f_s$; finally, for a pair of complex conjugate zeros a_s, $a_{\bar{s}}$ we set $\varphi_s := \operatorname{Re} f_s$, $\varphi_{\bar{s}} := \operatorname{Im} f_{\bar{s}}$.

Lemma 4. *The functions $\gamma_s^\pm$, $s = 1, \dots, g$; φ_j, $j = 1, \dots, l$, make up a local real analytic system of coordinates on the stratum* (13).

Proof. The positions of the branch points $e_1, \dots, e_{2g}$ of the curve M and the zeros $a_1, \dots, a_l$ of the differential $d\xi$ are complex-valued functions of local coordinates on the stratum. The differentials of the new coordinate functions have the following expressions:

$$
\begin{aligned}
d\gamma_s^+ &= -\Bigg(\sum_{j=1}^{2g}\bigg[\Big(\varepsilon_j - \frac{1}{2}\Big)\int_{C_s^+}\frac{d\xi}{x-e_j}\bigg]\,de_j + \sum_{j=1}^{l}\bigg[\alpha_j\int_{C_s^+}\frac{d\xi}{x-a_j}\bigg]\,da_j\Bigg),\\
&\qquad s = 1, \dots, g,\\
d\gamma_s^- &= i\Bigg(\sum_{j=1}^{2g}\bigg[\Big(\varepsilon_j - \frac{1}{2}\Big)\int_{C_s^-}\frac{d\xi}{x-e_j}\bigg]\,de_j + \sum_{j=1}^{l}\bigg[\alpha_j\int_{C_s^-}\frac{d\xi}{x-a_j}\bigg]\,da_j\Bigg),\\
&\qquad s = 1, \dots, g,\\
d\varphi_s &= -\begin{bmatrix}\mathrm{Re}\\ \text{or}\\ \mathrm{Im}\end{bmatrix}\Bigg(\sum_{j=1}^{2g}\bigg[\Big(\varepsilon_j - \frac{1}{2}\Big)\int_{F_s}\frac{d\xi}{x-e_j}\bigg]\,de_j + \sum_{j=1}^{l}\bigg[\alpha_j\int_{F_s}\frac{d\xi}{x-a_j}\bigg]\,da_j\Bigg),\\
&\qquad s = 1, \dots, l.
\end{aligned}
$$

If they are linearly independent, then there exists on M a non-trivial *real* differential

$$
d\omega = \Bigg(\sum_{j=1}^{2g}\frac{E_j}{x-e_j} + \sum_{j=1}^{l}\frac{A_j}{x-a_j}\Bigg)d\xi,
$$

with constants E_j and A_j such that all integrals over the cycles $C_s^\pm$ vanish, as well as the real or imaginary (in accordance with the definition of φ_s) parts of the integrals over the paths F_s. The real symmetry $\overline{\int_{F_s} d\omega} = \int_{\bar{J}F_s} d\omega$ and the above relations between $\bar{J}F_s$ and F_s yield the relations $\int_{F_s} d\omega = 0$, $s = 1, \dots, l$.

The poles of $d\omega$ can be located only at ramification points of the curve M, and the residues at these poles are zero because $d\omega$ is antisymmetric with respect to the hyperelliptic involution. Since the cyclic and the polar periods of $d\omega$ alike are equal to zero, the Abelian integral $\omega(x,w) := \int_{(e_{2g+2},0)}^{(x,w)} d\omega$ is a single-valued function on M. The integral ω is also antisymmetric with respect to the involution of the curve, therefore the equalities $\int_{F_s} d\omega = 0$ mentioned above mean that ω vanishes at the points in M lying over the a_s, $s = 1,\dots,l$. The even function $w\omega$ has a unique singularity, a pole at infinity, and therefore it is a polynomial in x. The degree of the polynomial $w\omega$ is at most $g+1$, and it has $g+2$ zeros with multiplicities taken into account: the zeros e_j of multiplicities ε_j for $j = 1,\dots,2g$ and multiplicity $1+\varepsilon_j$ for $j = 2g+1, 2g+2$, and the zeros a_j of multiplicities α_j, $j = 1,\dots,l$. Hence $d\omega = 0$ and the differentials of the real analytic functions $\gamma_s^{\pm}$, $s = 1,\dots,g$; φ_j, $j = 1,\dots,l$, are linearly independent on the stratum (13).

To complete the proof of Theorem 6 we unravel the situation of the projections of fibres of partial period maps onto the base. On a fixed curve M there exists a unique real differential $d\xi$ with residue -1 at ∞_+ and with prescribed real periods $\gamma_1^+,\dots,\gamma_g^+$ (or with prescribed imaginary periods $i\gamma_1^-,\dots,i\gamma_g^-$). This produces a bijection of the fibres $\Pi_\pm$ onto the base $\widetilde{\mathcal{H}}_g^k$, which is a cell. The non-degeneracy (= the maximum possible rank) of the restriction of the projection $\mathcal{N} \to \widetilde{\mathcal{H}}_g^k$ to the fibre $\Pi_\pm$ follows from the infinitesimal version of these arguments. Indeed, a vertical vector tangent to $\mathcal{N}$ at the point $d\xi$ is identified with a holomorphic real differential $d\omega$ on the curve M carrying $d\xi$. If this vector is tangent to the fibre of the period map, then all the integrals of $d\omega$ over the cycles C_s^+ (respectively, over the cycles C_s^-) $s = 1,\dots,g$, vanish. Hence $d\omega$, and therefore also the vertical tangent vector to the fibre $\Pi_\pm$, are trivial.

4.4. Period map on the moduli space. For the study of Abel's equations (12) we identify the submanifold of $\Omega^1\widetilde{\mathcal{H}}_g^k$ consisting of the differentials $d\eta_M$ associated with curves M and the base of this vector bundle. We know from Theorem 6(1) that the manifold $\mathcal{N} \cap (\Pi_+)^{-1}(0)$ projects onto $\widetilde{\mathcal{H}}_g^k$ without singularities. Now, the partial period map Π_- is defined directly on the universal covering space of the moduli space. On it, similarly to the cohomology $H^1(M_0)$, we have the action of the braid group on $g-k+1$ strings (see § 4.2).

Lemma 5. *The period map $\Pi_- : \widetilde{\mathcal{H}}_g^k \to (H_1^-(M_0,\mathbb{R}))^*$ is interchangeable with the action of Br_{g-k+1}.*

Proof. If an element C of the homology fibration $H_1^-\mathcal{H}_g^k$ projects to the initial point of a path τ in the base $\mathcal{H}_g^k$, then we can perform a parallel translation of C along this path using the natural flat connection. We denote the result of this translation by $C\cdot\tau$; this action of paths on cycles is associative: $C\cdot(\tau_1\tau_2) = (C\cdot\tau_1)\cdot\tau_2$, and depends only on the homology class of the path. Points in the universal covering space $\widetilde{\mathcal{H}}_g^k$ are homotopy classes of paths $\tau \subset \mathcal{H}_g^k$ starting at the distinguished point M_0. The braid group $\mathrm{Br}_{g-k+1} \cong \pi_1(\mathcal{H}_g^k, M_0) \ni [\beta]$ acts on them by covering transformations $[\tau] \to [\beta\tau]$. The assertion of the lemma follows from the chain of the equalities

$$\langle \Pi_-([\beta]\cdot[\tau]) \mid C\rangle := -i\int_{C\cdot(\beta\tau)} d\eta_M = -i\int_{(C\cdot\beta)\cdot\tau} d\eta_M =: \langle [\beta]\cdot\Pi_-([\tau]) \mid C\rangle.$$

We shall call the inverse image of a functional $C_* \in (H_1^-(M_0,\mathbb{R}))^*$ in the universal covering space $\widetilde{\mathcal{H}}_g^k$ the manifold $\mathbb{T}(C_*)$. For instance, Abel's equations (12) define locally such a manifold for the functional C_* defined on the basis of the lattice of odd 1-cycles in M by the equalities $\langle C_* \mid C_s^- \rangle = 2\pi m_s/n$, $s = 0,1,\dots,g$. We now list the properties of these manifolds that we know already from Theorems 5, 6 and Lemma 5.

Theorem 7. (1) $\mathbb{T}$ *is a smooth g-dimensional submanifold of* $\widetilde{\mathcal{H}}_g^k$.

(2) *Two $\mathbb{T}$-manifolds are either disjoint or coincide.*

(3) $\mathbb{T}(\beta\cdot C_*) = \beta\cdot\mathbb{T}(C_*)$, $C_* \in \Pi_-(\widetilde{\mathcal{H}}_g^k)$, $\beta \in \mathrm{Br}_{g-k+1}$.

(4) *Points in the universal cover $\widetilde{\mathcal{H}}_g^k$ associated with polynomials of degree n fill $\mathbb{T}$-manifolds corresponding to functionals in the inverse lattice* $4\pi n^{-1}(L_{M_0})^*$.

(5) *The 'rational' manifolds* $\mathbb{T}(C_*)$, $C_* \in 2\pi(H_1^-(M_0,\mathbb{Q}))^*$, *corresponding to various polynomials are dense in* $\widetilde{\mathcal{H}}_g^k$.

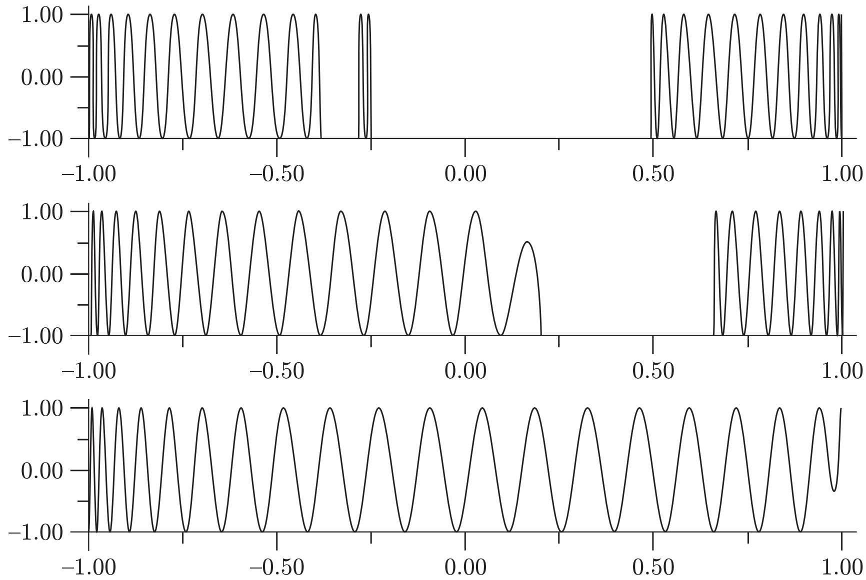

Figure 2. Extremal polynomials $P_{50}(x)$ for $g=2$, $k=3,2,1$

The following questions arise in the further study of the period map of the moduli space.

(1) Find the image of $\widetilde{\mathcal{H}}_g^k$ under the partial period map Π_-. It was shown in [12] that the image of the universal covering space for $k = g+1$ is the interior of a g-simplex; for $k = g$ it is a union of k open g-simplexes, and for $k < g$ it is a countable union of open g-simplexes indexed by braids.

(2) Describe the topology of a $\mathbb{T}$-manifold. As shown in [10], it is always a cell for $k = g+1$. Using the methods of [12] one can obtain a cell decomposition of a $\mathbb{T}$-manifold and the question of its topology reduces to combinatorics. However,

to describe the topology knowing the neighbourhood relations between cells in the decomposition requires hard work even for small g. Our calculations for $g = 1, 2, 3$ and all $k = 0, \dots, g + 1$ show that in these cases $\mathbb{T}$ is also a cell, whatever the functional in the range of Π_- might be. Probably, the following result holds.

Conjecture. *A* $\mathbb{T}$*-manifold is always a cell.*

(3) Is the $\mathbb{T}$-manifold folded by the projection onto the moduli space $\mathcal{H}_g^k$? In other words, has the action of the braid group on the functionals in $(H_1^-(M_0, \mathbb{R}))^*$ fixed points in the range of the period map? In [12] the reader can find an example of a fibre of this map that is invariant under the action of a certain braid. It is likely that the topology of a $\mathbb{T}$-manifold can change after the projection onto the moduli space $\mathcal{H}_g^k$.

(4) Learn to effectively solve Abel's equations (12), that is, to find the position of a fixed $\mathbb{T}$-manifold in the space $\widetilde{\mathcal{H}}_g^k$. The latter can be realized as a subdomain of the Euclidean space with points corresponding to the generators of Schottky groups of a special type. Similar calculations are the subject of [11]. The graphs of three 2-extremal polynomials of degree $n = 50$ obtained by computer calculations are plotted in Fig. 2.

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