PS_3 integral equations and projective structures on Riemann surfaces

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PS_3 integral equations and projective structures on Riemann surfaces

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Abstract. A complex-geometric theory of the Poincaré–Steklov integral equation is developed. Solutions of this equation are effectively represented and its spectrum is localized.

Bibliography: 15 titles.

§ 1. Origins of the PS equation and a survey of results

An arbitrary non-singular change of variable $R(t)$ on the closed interval $[-1,1]$ can play the role of the functional parameter in the (Poincaré–Steklov) singular integral equation of the following form:

$$\lambda\,\mathrm{V.p.}\int_I \frac{u(t)}{t-x}\,dt - \mathrm{V.p.}\int_I \frac{u(t)\,dR(t)}{R(t)-R(x)} = \mathrm{const}, \qquad x \in I := (-1,1), \tag{1}$$

where λ is the spectral parameter, $u(t)$ is the unknown function, and const is an unknown constant. The problem consists in finding eigenvalue-eigenvector pairs $(\lambda, u(t))$ of equation (1) with non-trivial function $u(t)$ from a prescribed function class.

The subject of the present paper is the Poincaré–Steklov equations with parameter $R_3(t)$ that is a rational function of degree 3; we call them in what follows Poincaré–Steklov equations of degree 3 or PS_3. The main result of the paper is the establishment of a one-to-one correspondence between eigenvalue-eigenvector pairs of PS_3 and linearly polymorphic functions (= projective structures [1]) on Riemann surfaces, which can be effectively calculated in terms of R_3. This result enables us to find explicit geometric representations for eigenvalue-eigenvector pairs of (1). As a by-product, a localization of the spectrum of the PS_3 equations is obtained as a consequence of constraints on the monodromy of projective structures.

In the numerical solution of boundary-value problems for the Laplace equation one uses the method of domain partitioning: the domain Ω in which the solution is sought is partitioned into smaller subdomains by artificial inner boundaries. At these artificial boundaries one sets arbitrary Dirichlet data and solves the resulting boundary-value problems in the subdomains. Of course, the solutions do not combine together into a solution of the original problem because the normal derivatives

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of the solutions at the inner boundaries make jumps. At the second step of the procedure one sets new boundary conditions depending on the discrepancy of the normal derivatives of the solutions obtained at the first step. The procedure is then repeated until the jumps of the normal derivatives of the solutions become small. For the proof of convergence and for the optimization of this iterative procedure in concrete cases one requires information about the spectrum of the problem below.

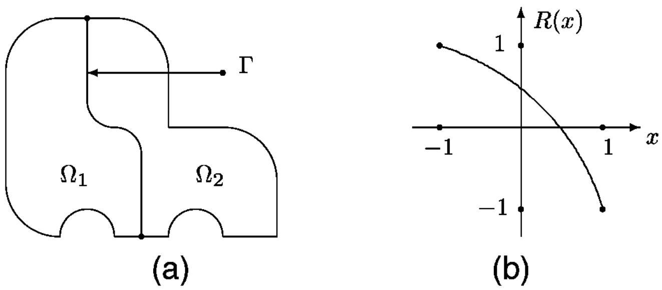


Figure 1. (a) Partitioning of the solution domain; (b) change of variable $R\colon I \to I$

Let Γ *be a smooth arc partitioning a plane domain* Ω *into subdomains* Ω_1 *and* Ω_2 (*see Fig.* 1(a)). *Consider the eigenvalue problem*

$$\begin{aligned}
&\Delta U_1 = 0 \text{ in } \Omega_1; \quad U_1 = 0 \text{ at } \partial\Omega_1 \setminus \Gamma;\\
&\Delta U_2 = 0 \text{ in } \Omega_2; \quad U_2 = 0 \text{ at } \partial\Omega_2 \setminus \Gamma;\\
&U_1 = U_2 \text{ at } \Gamma;\\
&-\lambda\frac{\partial U_1}{\partial n} = \frac{\partial U_2}{\partial n} \text{ at } \Gamma;
\end{aligned} \tag{2}$$

here Δ *is the Laplace operator*, λ *is the spectral parameter*, *and* n *is the normal to* Γ. The same spectral problem arises in the analysis of the stability of the free interface between two fluids (water–oil) when one of them is displaced in a porous medium (sand). Similar problems with spectral parameter in the boundary conditions for one domain were considered by Poincaré and Steklov about a century ago.

The traces of eigenfunctions of (2) at the inner boundary Γ satisfy a certain Poincaré–Steklov equation (1). For if the normal derivatives at Γ of the harmonic functions U_1 and U_2 are proportional with coefficient $-\lambda$, then the tangential (to Γ) derivatives of the conjugate harmonic functions V_1 and V_2 are also related by the coefficient $-\lambda$. Integrating along Γ we obtain

$$\lambda V_1(y) + V_2(y) = \text{const}, \qquad y \in \Gamma. \tag{3}$$

If the domain Ω_k ($k = 1, 2$) is a half-plane, then the boundary values of the V_k are the Hilbert transforms of the boundary values of the U_k. In the case of arbitrary (simply connected) domains Ω_1 and Ω_2 one can consider their conformal maps ω_1 and ω_2 onto the upper half-plane normalized by the conditions $\omega_k(\Gamma) = I$, $k = 1, 2$. Equation (3) can now be written as follows:

$$-\frac{\lambda}{\pi}\,\text{V.p.}\int_I \frac{U_1(\omega_1^{-1}t)}{t - \omega_1 y}\,dt - \frac{1}{\pi}\,\text{V.p.}\int_I \frac{U_2(\omega_2^{-1}t')}{t' - \omega_2 y}\,dt' = \text{const}, \qquad y \in \Gamma.$$

Introducing the new notation

$$\begin{aligned} x &:= \omega_1 y \in I; \\ R &:= \omega_2 \circ \omega_1^{-1} : I \to \Gamma \to I; \\ u(t) &:= U_1(\omega_1^{-1} t) \end{aligned}$$

and making the change of variable $t' = R(t)$ in the second integral we arrive at the Poincaré–Steklov equation (1). We point out that the function $R(x)$ is decreasing in this case.

The boundary-value problem (2) in the Sobolev space W_2^1 has been considered in [2]–[4]. It is shown in [2] that the spectrum of this problem is concentrated on the positive semi-axis; its upper and lower bounds are also found in that paper and a model problem is solved by separation of variables. In [4] this author investigated the case of a smooth arc Γ whose tangent at each end-point of Γ is the bisector of the angle between the one-sided tangents to the boundary $\partial\Omega$. Under the above assumptions the spectrum of (2) is discrete; it has a unique accumulation point $\lambda = 1$; the quantity $\sum_{\lambda \in Sp} |\lambda - 1|^2$ is finite and admits an effective estimate; and the traces at Γ of the eigenfunctions of the problem make up an orthogonal basis in the Sobolev space $\widetilde{W}_2^{1/2}(\Gamma)$. By contrast, if Γ is not smooth or the tangent to Γ is not a bisector at some end-point, then there exists an interval filled with continuous spectrum, which can be explicitly calculated [3].

Attempts to solve the PS integral equation in closed form for concrete analytic functions $R(t)$ [5] have led to the discovery of a close connection between equations with rational parameter R_n and the classical problem posed by Riemann [6], [7]: *find a collection of n analytic functions on the Riemann sphere with punctures $a_1, \dots, a_p$, with at most power growth in the neighbourhood of the points a_k such that on going around each of the punctures they undergo a prescribed linear substitution with constant coefficients.*

The idea of the replacement of the PS equation by the Riemann monodromy problem is that in principle there arises the opportunity to reduce the spectral problem for an integral equation to a *finite system of transcendental equations.* These equations connect the so-called accessory parameters and the position of the 'dummy poles' of a Fuchsian ODE of order n with its monodromy [6]. This approach enables one to find effective representations and, in particular, explicit formulae [5], [8] for eigenvalue-eigenvector pairs of the integral equation.

This paper is organized as follows: in the next section we discuss in detail the connections existing between the PS equation with rational parameter R and the Riemann problem. In § 3 we find the monodromies corresponding to various rational functions R_3 of degree 3. These monodromies are always locally finite, so that the solution W of the Riemann problems becomes analytic when lifted onto the compact Riemann surface $\mathcal{M}(R_3)$ selected in § 4. In § 5 we observe that the monodromy matrices in our problem are (pseudo-)orthogonal and therefore the solution W ranges in a (possibly degenerate) quadric in $\mathbb{C}^3$. A non-degenerate projective quadric is isomorphic to $\mathbb{CP}_1 \times \mathbb{CP}_1$, therefore a solution of the monodromy problem defines a pair of multivalued meromorphic functions p^+ and p^- on $\mathcal{M}(R_3)$.

These functions turn out to be linearly polymorphic, that is, they transform linear fractionally as the independent variable proceeds along cycles on $\mathcal{M}$. Linearly polymorphic functions have a geometric representation as so-called membranes; this goes back to Klein's work and describes in effect the eigenvalue-eigenvector pairs of the PS_3 integral equation. In the last section we find bounds for the spectrum of the PS_3 equation using the fact that the monodromy of the projective structure is not unitary.

§ 2. PS_n equations and the Riemann monodromy problem

We shall seek *Hölder* solutions u of the PS_n equation in which $R = R_n$ is a rational function of degree n defining a non-singular[1] change of variable on $[-1,1]$, that is,

$$\frac{d}{dt}R_n(t) \neq 0, \qquad t \in [-1,1]. \tag{4}$$

For fixed $x \in I$ we expand the kernel of the second integral in (1) in a sum of elementary fractions:

$$\frac{\dot{R}_n(t)}{R_n(t) - R_n(x)} = \sum_{k=1}^{n} \frac{1}{t - z_k(x)} - \frac{\dot{Q}(t)}{Q(t)}, \tag{5}$$

where $Q(t)$ is the denominator of the function $R_n(t)$ represented as a non-cancellable ratio of polynomials; $z_1(x) = x, z_2(x), \dots, z_n(x)$ are all the solutions of the equation $R_n(z_k) = R_n(x)$ (including the multiple ones and the one equal to ∞). This expansion abounds with Cauchy kernels, which allows us to write integral equation (1) in terms of the Cauchy-type integral

$$\Phi(x) := \int_I \frac{u(t)}{t - x}\, dt + \mathrm{const}', \qquad x \in \widehat{\mathbb{C}} \setminus \overline{I}, \tag{6}$$

so that the solution $u(t)$ can be subsequently recovered by the Sokhotskiĭ–Plemelj formula:

$$u(t) = (2\pi i)^{-1}[\Phi(t + i0) - \Phi(t - i0)], \qquad t \in I. \tag{7}$$

The constant const', which we set here to be

$$\mathrm{const}' := \frac{1}{\lambda - n}\left[\int_I \frac{u(t)\dot{Q}(t)}{Q(t)}\, dt - \mathrm{const}\right], \tag{8}$$

is added to compensate for the constants arising in the substitution of (6) in (1). In this way we prove the following result.

Lemma 1 [8]. *For $\lambda \neq 1, n$ the transformations* (6) *and* (7) *bring about a one-to-one correspondence between eigenvalue-eigenvector pairs* $(\lambda, u(t))$ *of the* PS_n *equations and the holomorphic* (*in* $\widehat{\mathbb{C}} \setminus \overline{I}$) *non-trivial solutions* $\Phi(x)$ *of the functional*

[1] Non-singularity condition (4) means that Γ satisfies the above-stated assumptions, which ensures that the problem (2) has a discrete spectrum.

equation

$$\Phi^+(x) + \Phi^-(x) = \delta \sum_{k=2}^{n} \Phi(z_k(x)), \qquad x \in I, \tag{9}$$

$$\delta = \frac{2}{\lambda - 1}, \tag{10}$$

with boundary values $\Phi^\pm(x) := \Phi(x \pm i0)$ *satisfying the Hölder condition at* $\overline{I}$.

2.1. Motivation for introduction of monodromy. Functional equation (9) enables one to find the maximal domain of holomorphy of the function $\Phi(x)$, which is originally defined in the complement of the cut I. In fact, non-singularity condition (4) means that $R_n(x)$ is univalent in some neighbourhood $\mathcal{U} \supset I$. If $x = z_1(x)$ is a point in this neighbourhood, then the other points $z_2(x), \dots, z_n(x)$ lie outside $\mathcal{U}$, that is, in the holomorphy domain of Φ. Hence the right-hand side of (9) is a holomorphic function of x in $\mathcal{U}$ (although some terms in this sum may have branch points in $\mathcal{U}$, the sum as a whole is holomorphic). Now, the equality

$$\Phi^+(x) = -\Phi^-(x) + \delta \sum_{k=2}^{n} \Phi(z_k(x))$$

defines an analytic continuation of Φ^+ downwards, across the cut. In particular, the analytic continuation of the germ of $\Phi^\pm$ across the cut, along a path from one bank of the cut to the other bank, gives us $\Phi^\mp$. In other words in a small neighbourhood of I the function Φ is holomorphic on the Riemann surface of $\sqrt{1 - x^2}$.

To investigate the global domain of holomorphy of $\Phi(x)$ it is useful to consider the (only locally defined) vector $W(x) := (\Phi(z_1(x)), \Phi(z_2(x)), \dots, \Phi(z_n(x)))^t$. It may fail to be analytic for two reasons: (1) at points $x \in R_n^{-1}(\pm 1)$ the variable $z_k(x)$ corresponding to some component of the vector W takes one of the values ± 1, which are the branch points of Φ, or (2) at a branch point of one of the algebraic functions $z_k(x)$ the corresponding component of the vector also branches. On going around points of the first type one of the components of the vector is transformed — in accordance with (9) — into a linear combination of the components; for instance, if $x = \pm 1$, then W_1 is replaced by $-W_1 + \delta \sum_{j=2}^{n} W_j$. The remaining components may rearrange at worst. On going around points of the second type we arrive at a vector with rearranged components. Thus, the vector W constructed from a solution of the PS_n integral equation solves some Riemann monodromy problem.

2.2. Monodromy of $\mathbf{PS}_n$ equation. Our first aim is to find the monodromy $\mathbf{T}$ corresponding to the Poincaré–Steklov equation of degree n. It can be calculated as a modification of the monodromy $\mathbf{T}_*$ of the n-sheeted cover $R_n(x) \colon \mathbb{CP}_1 \to \mathbb{CP}_1$.

Let $a_1, \dots, a_p$ be the critical values of R_n; they can include the points ± 1, which require special attention (see Fig. 2(a)) because they are the projections of the ramification points of Φ. We define both monodromies on the punctured sphere $\mathcal{Y} := \mathbb{CP}_1 \setminus \{a_1, \dots, a_p, -1, 1\}$. The cover of the base $\mathcal{Y}$ by the space $\mathcal{X} := R_n^{-1}(\mathcal{Y})$ is non-ramified, therefore one can define in the standard manner [9] a representation $\mathbf{T}_*$ from $\pi_1(\mathcal{Y})$ into the symmetric group of degree n (the monodromy of the cover). Namely, let $y_0 \in \mathcal{Y}$ be the base point of the fundamental group,

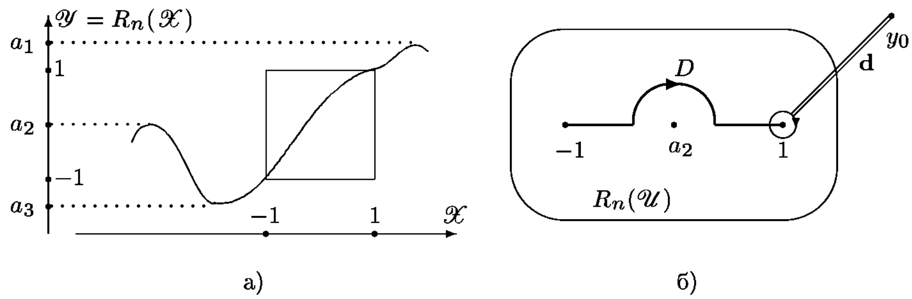


Figure 2. (a) Branch points of the cover R_n; (b) selection of the deformation of the cut I

$x_1, x_2, \dots, x_n$ the points in $\mathcal{X}$ lying over it, and let $\mathbf{r} \subset \mathcal{Y}$ be a loop with initial point y_0. Then there exists a unique permutation matrix $\mathbf{T}_*$ such that

$$\mathbf{T}_*([\mathbf{r}]) \cdot (x_1, \dots, x_n)^t := (x_1 \cdot \mathbf{r}, x_2 \cdot \mathbf{r}, \dots, x_n \cdot \mathbf{r})^t. \tag{11}$$

Here the path $\mathbf{r}$ on the base acts on a point x in the covering space if x projects onto the initial point of $\mathbf{r}$; we denote by $x \cdot \mathbf{r}$ the end-point of the lift of $\mathbf{r}$ to the covering space with initial point x. This action is associative: $(x \cdot \mathbf{r}_1) \cdot \mathbf{r}_2 = x \cdot (\mathbf{r}_1 \cdot \mathbf{r}_2)$. The group property of the map $\mathbf{T}_*$ is ensured by the following result.

Lemma 2 [8]. *Let $\mathbf{s}$ be a path and $\mathbf{r}$ a loop in $\mathcal{Y}$, both with initial point y_0. Then*

$$\mathbf{T}_*([\mathbf{r}])(x_1 \cdot \mathbf{s}, x_2 \cdot \mathbf{s}, \dots, x_n \cdot \mathbf{s})^t = (x_1 \cdot \mathbf{rs}, x_2 \cdot \mathbf{rs}, \dots, x_n \cdot \mathbf{rs})^t. \tag{12}$$

The monodromy $\mathbf{T}\colon \pi_1(\mathcal{Y}) \to GL_n(\mathbb{C})$ depends on the spectral parameter λ. For its calculation we consider a simple arc D from the puncture -1 to the puncture 1 that lies entirely in the neighbourhood $R_n(\mathcal{U})$ of the interval I and avoids the punctures of $\mathcal{Y}$. We take for D the interval I in the case when it does not contain critical values of R_n, or a small deformation of it otherwise (see Fig. 2). We fix a 'lasso' $\mathbf{d}$ encircling one of the points ± 1 and intersecting D once — for definiteness, from the right to the left. Then the following result distinguishes a point in the set $\{x_1, \dots, x_n\} := R_n^{-1}(y_0)$.

Lemma 3 [8]. *The inverse image $R_n^{-1}(y_0)$ contains a unique point x_1 such that the lift to $\mathcal{X}$ of the loop $\mathbf{d}$ starting at this point is a lasso encircling one of the points ± 1.*

Having fixed the cut D and the loop $\mathbf{d}$ intersecting it at a single point we obtain a decomposition of the fundamental group into free generators:

$$\pi_1(\mathcal{Y}, y_0) = \langle \pi_1(\mathcal{Y} \setminus D, y_0), [\mathbf{d}] | \varnothing \rangle\,. \tag{13}$$

We define the restrictions of the representation $\mathbf{T}$ to these generators:

$$\mathbf{T}([\mathbf{d}]) := \mathbf{D}\mathbf{T}_*([\mathbf{d}]) = \mathbf{T}_*([\mathbf{d}])\mathbf{D}; \qquad \mathbf{T}([\mathbf{r}]) := \mathbf{T}_*([\mathbf{r}]), \quad [\mathbf{r}] \in \pi_1(\mathcal{Y} \setminus D, y_0), \tag{14}$$

where

$$\mathbf{D} := \left\| \begin{array}{cccccc} -1 & \vdots & \delta & \delta & \dots & \delta \\ \dots & \dots & \dots & \dots & \dots & \dots \\ & \vdots & 1 & & & \\ & \vdots & & 1 & & \\ & \vdots & & & \ddots & \\ & \vdots & & & & 1 \end{array} \right\| \in GL_n(\mathbb{C}); \tag{15}$$

we indicate only the entries of the matrix that are distinct from zero. The matrix $\mathbf{D}$ commutes with each permutation matrix fixing the first element and, in particular, with $\mathbf{T}_*([\mathbf{d}])$, as follows by Lemma 3. The square of $\mathbf{D}$ is the identity matrix.

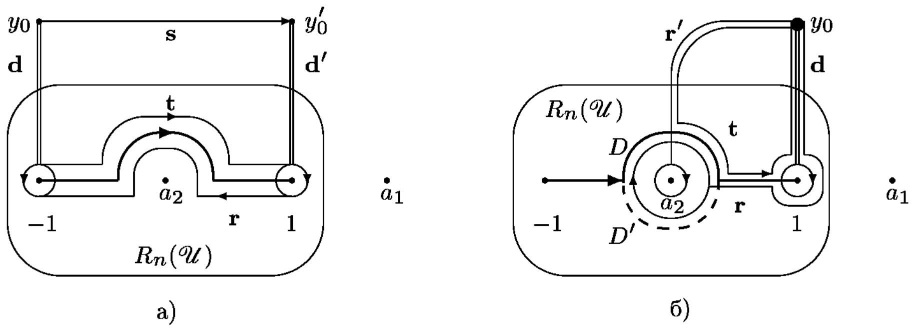


Figure 3. (a) Decomposition of the lasso $\mathbf{s}\mathbf{d}'\mathbf{s}^{-1}$ into generators; (b) lasso $\mathbf{r}'$ and its decomposition into generators

2.3. Ambiguity in definition of monodromy. We show below that the freedom in our choice of the cut D, the 'lasso' $\mathbf{d}$, the base point y_0, and the numbering of the covering points $x_2, x_3, \dots, x_n$ results — at worst — in the conjugation of the representation $\mathbf{T}$ by a permutation matrix.

For a fixed cut D we consider two suitable collections of a lasso, a base point, and points covering the base point: $\{\mathbf{d}; y_0; x_1, \dots, x_n\}$ and $\{\mathbf{d}'; y_0'; x_1', \dots, x_n'\}$. We assume that the points x_1 and x_1' are the ones described by Lemma 3. The following result relates the pairs of monodromies $\mathbf{T}_*$, $\mathbf{T}_*'$ and $\mathbf{T}$, $\mathbf{T}'$ corresponding to these collections.

Lemma 4. *Let* $\mathbf{s}$ *be a path from* y_0 *to* y_0' *disjoint from* D*. Let* $\mathbf{K}$ *be the permutation matrix such that*

$$\mathbf{K} \cdot (x_1 \cdot \mathbf{s}, x_2 \cdot \mathbf{s}, \dots, x_n \cdot \mathbf{s})^t := (x_1', \dots, x_n')^t.$$

Then the monodromies corresponding to distinct collections are conjugate:

$$\begin{aligned} \mathbf{T}_*'([\mathbf{s}^{-1}\mathbf{r}\mathbf{s}]) &= \mathbf{K}\mathbf{T}_*([\mathbf{r}])\mathbf{K}^{-1}, \\ \mathbf{T}'([\mathbf{s}^{-1}\mathbf{r}\mathbf{s}]) &= \mathbf{K}\mathbf{T}([\mathbf{r}])\mathbf{K}^{-1}, \end{aligned} \qquad [\mathbf{r}] \in \pi_1(\mathcal{Y}, y_0). \tag{16}$$

Proof. For the monodromy of the covering we have the chain of equalities

$$\begin{aligned}\mathbf{T}'_*([\mathbf{s}^{-1}\mathbf{rs}])(x'_1,\dots,x'_n)^t &:= (x'_1\cdot\mathbf{s}^{-1}\mathbf{rs}, x'_2\cdot\mathbf{s}^{-1}\mathbf{rs}\dots,x'_n\cdot\mathbf{s}^{-1}\mathbf{rs})^t\\ &= \mathbf{K}(x_1\cdot\mathbf{rs},x_2\cdot\mathbf{rs},\dots,x_n\cdot\mathbf{rs})^t\\ &= \mathbf{KT}_*([\mathbf{r}])\mathbf{K}^{-1}(x'_1,\dots,x'_n)^t.\end{aligned}$$

We verify the second equality in (16), which connects $\mathbf{T}$ and $\mathbf{T}'$, for the free generators (13). It has already been proved for the subgroup $\pi_1(\mathcal{Y}\setminus D,y_0)$ and it remains to calculate $\mathbf{T}'([\mathbf{s}^{-1}\mathbf{ds}])$. We factor the loop $\mathbf{sd}'\mathbf{s}^{-1}$ in a product of the generators (13):

$$\mathbf{sd}'\mathbf{s}^{-1}=\mathbf{rdt},\qquad [\mathbf{r}],[\mathbf{t}]\in\pi_1(\mathcal{Y}\setminus D,y_0).$$

The construction of the loops $\mathbf{r}$ and $\mathbf{t}$ is clear from Fig. 3(a). We note the following equalities, which hold by the construction of the loops:

$$x_1\cdot\mathbf{r}^{-1}\mathbf{s}=x'_1,\qquad x_1\cdot\mathbf{ts}=x'_1. \tag{17}$$

We have the following calculation:

$$\begin{aligned}\mathbf{T}'([\mathbf{s}^{-1}\mathbf{ds}]) &= \mathbf{T}'([\mathbf{s}^{-1}\mathbf{r}^{-1}\mathbf{s}]\cdot[\mathbf{d}']\cdot[\mathbf{s}^{-1}\mathbf{t}^{-1}\mathbf{s}])\\ &:= \mathbf{KT}_*([\mathbf{r}^{-1}])\mathbf{K}^{-1}\cdot\mathbf{D}\cdot\mathbf{KT}_*([\mathbf{rdt}])\mathbf{K}^{-1}\cdot\mathbf{KT}_*([\mathbf{t}^{-1}])\mathbf{K}^{-1}\\ &\overset{(\star)}{=} \mathbf{KDT}_*([\mathbf{d}])\mathbf{K}^{-1} =: \mathbf{KT}([\mathbf{d}])\mathbf{K}^{-1}.\end{aligned}$$

In the equality marked by $(\star)$ we interchanged the factors $\mathbf{T}_*([\mathbf{r}^{-1}])\mathbf{K}^{-1}$ and $\mathbf{D}$ because the permutation corresponding to the former leaves the first element fixed:

$$\begin{aligned}\mathbf{T}_*([\mathbf{r}^{-1}])\mathbf{K}^{-1}\cdot(x'_1,\dots,x'_n)^t &= \mathbf{T}_*([\mathbf{r}^{-1}])\cdot(x_1\cdot\mathbf{s},\dots,x_n\cdot\mathbf{s})^t\\ &= (x_1\cdot\mathbf{r}^{-1}\mathbf{s},\dots,x_n\cdot\mathbf{r}^{-1}\mathbf{s})^t \overset{(17)}{=} (x'_1,*,\dots,*)^t.\end{aligned}$$

We now study the influence of variations of the cut D on the monodromy $\mathbf{T}$. For two cuts, D and D', we select the same base point y_0, the lasso $\mathbf{d}$, and the points $x_1,\dots,x_n\in R_n^{-1}(y_0)$. This gives us monodromies $\mathbf{T}$ and $\mathbf{T}'$.

Lemma 5. *The monodromies $\mathbf{T}$ and $\mathbf{T}'$ are the same.*

Proof. We consider an elementary modification $D\to D'$ of the cut: the replacement of the left detour of the puncture $a_k\in I$ by the right one (see Fig. 3(b)). An arbitrary modification of the cut is a composite of these elementary modifications.

Consider now a lasso $\mathbf{r}'$ encircling the point a_k and disjoint from D'. We have the following decomposition of the fundamental group into (free) generators:

$$\pi_1(\mathcal{Y})=\langle\pi_1(\mathcal{Y}\setminus D)\cap\pi_1(\mathcal{Y}\setminus D'),[\mathbf{d}],[\mathbf{r}']|\varnothing\rangle\,.$$

For the first two generators the equality $\mathbf{T}=\mathbf{T}'$ is a direct consequence of the definition. To verify that $\mathbf{T}([\mathbf{r}'])=\mathbf{T}'([\mathbf{r}'])$ we factor $[\mathbf{r}']$ in a product of the generators (13):

$$\mathbf{r}'=\mathbf{td}^{-1}\mathbf{rdt}^{-1},\qquad [\mathbf{r}],[\mathbf{t}]\in\pi_1(\mathcal{Y}\setminus D).$$

The constructions of the loop $\mathbf{t}$ and the lasso $\mathbf{r}$ are clear from Fig. 3(b); in particular, we point out the equality

$$x_1 \cdot \mathbf{r} = x_1. \tag{18}$$

We have the following equalities:

$$\begin{aligned}\mathbf{T}([\mathbf{r}']) &:= \mathbf{T}_*([\mathbf{t}\mathbf{d}^{-1}])\mathbf{D}\mathbf{T}_*([\mathbf{r}])\mathbf{D}\mathbf{T}_*([\mathbf{d}\mathbf{t}^{-1}]) \\ &\overset{(\star)}{=} \mathbf{T}_*([\mathbf{t}\mathbf{d}^{-1}\mathbf{r}\mathbf{d}\mathbf{t}^{-1}]) = \mathbf{T}_*([\mathbf{r}']) =: \mathbf{T}'([\mathbf{r}']).\end{aligned}$$

In the equality $(\star)$ we use here the interchangeability of $\mathbf{D}$ and $\mathbf{T}_*([\mathbf{r}])$, which follows from (18), and also the relation $\mathbf{D} = \mathbf{D}^{-1}$.

2.4. Connection between the integral equality and the Riemann problem. The following result holds for the above-defined monodromy $\mathbf{T}$.

Theorem 1 [8]. *For $\lambda \notin \{1, n\}$ the eigenvalue-eigenvector pairs $(\lambda, u(t))$ of* PS_n *equation* (1) *are in one-to-one correspondence with non-trivial solutions W of the Riemann problem with monodromy* $\mathbf{T}$ *defined in* (14) *and* (15), *these solutions being analytic on $\mathcal{Y}$ and bounded in the neighbourhood of the punctures $\pm 1, a_1, \dots, a_p$; here δ is as in* (10).

Scheme of the proof. (1) For a fixed 'eigenpair' $(\lambda, u(t))$ of the PS_n equation we consider a solution W of the Riemann monodromy problem. It is not single-valued on the punctured sphere $\mathcal{Y}$ and can be defined in this way only on its universal cover $\widetilde{\mathcal{Y}}$. Let $\widetilde{y}_0 \in \widetilde{\mathcal{Y}}$ be a point over the base point $y_0 \in \mathcal{Y}$. Then points of the universal cover have the form $\widetilde{y} = \widetilde{y}_0 \cdot \mathbf{s}$, where $\mathbf{s} \subset \mathcal{Y}$ is some path.

For points $\widetilde{y} \in \widetilde{\mathcal{Y}}$ with projections lying outside the cut D we define the vector W by the formula

$$W(\widetilde{y}_0 \cdot \mathbf{s}) := \mathbf{T}([\mathbf{r}])\mathbf{T}_*{}^{-1}([\mathbf{r}])\bigl(\Phi(x_1 \cdot \mathbf{s}), \Phi(x_2 \cdot \mathbf{s}), \dots, \Phi(x_n \cdot \mathbf{s})\bigr)^t, \tag{19}$$

where Φ is the solution of functional equation (9) corresponding to the 'eigenpair' $(\lambda, u(t))$, $\mathbf{r} := \mathbf{s} \cdot \mathbf{t}^{-1}$ is a loop, and the path $\mathbf{t} \subset \mathcal{Y}$ is arbitrary but disjoint from D. This definition does not depend on one's choice of a path $\mathbf{t}$ completing $\mathbf{s}$ to a loop.

It turns out that the vector W has the same boundary values from the left and from the right on the components of the inverse image of D in $\widetilde{\mathcal{Y}}$ in view of the fact that Φ satisfies (9). Hence W extends to a holomorphic vector-valued function on the entire universal cover. Covering transformations multiply W by constant matrices:

$$W([\mathbf{r}] \cdot \widetilde{y}) = W(\widetilde{y}_0 \cdot \mathbf{r} \cdot \mathbf{s}) = \mathbf{T}([\mathbf{r}])W(\widetilde{y}), \qquad [\mathbf{r}] \in \pi_1(\mathcal{Y}, y_0), \quad \widetilde{y} := \widetilde{y}_0 \cdot \mathbf{s}, \quad \mathbf{s} \subset \mathcal{Y}. \tag{20}$$

(2) Conversely, let W be the solution of the Riemann problem defined on the universal cover $\widetilde{\mathcal{Y}}$. On the sphere with n cuts $\mathcal{X} \setminus R_n^{-1}(D)$ we consider the holomorphic function

$$\Phi(x) = \Phi(x_k \cdot \mathbf{s}) := W_k(\widetilde{y}_0 \cdot \mathbf{s}), \qquad k = 1, \dots, n; \tag{21}$$

here the path $\mathbf{s} \subset \mathcal{Y}$ is disjoint from D and $x_k \in R_n^{-1}(y_0)$. This definition is also independent of our choice of the path $\mathbf{s}$ and the index k.

On the component $R_n^{-1}(D)$ that is a small deformation of I the function Φ turns out to satisfy (9), while its jumps at the other $n-1$ cuts are equal to zero. The analytic continuation of Φ is holomorphic on $\mathbb{CP}_1 \setminus \overline{I}$ and satisfies (9) at the points in I. By Lemma 1 the function

$$u(t) = (2\pi i)^{-1}[\Phi(t+i0) - \Phi(t-i0)], \qquad t \in I, \tag{22}$$

and the quantity $\lambda := 1 + 2/\delta$ make up an eigenvector-eigenvalue pair of the PS_n equation.

(3) The correspondences in parts (1) and (2) of the proof are reciprocal.

In our derivation of the Poincaré–Steklov equation it is obvious that the change of variable $R(t)$ is not uniquely defined by the domains in which one poses the differential problem (2); however, it is unique to within taking the composite map with two linear fractional transformations of the kind

$$\Lambda_\alpha(t) := \frac{t+\alpha}{\alpha t + 1}, \qquad \alpha \in (-1, 1), \tag{23}$$

preserving the interval $[-1, 1]$. Of course, such transformations cannot influence the spectrum of the integral transform. This is also obvious from Theorem 1.

Corollary. *Let $(\lambda, u(t))$ be an eigenvalue-eigenvector pair of the PS_n equation with rational function R_n. Then $(\lambda, u \circ \Lambda_\alpha(t))$ is an eigenvalue-eigenvector pair of the equation with parameter $\Lambda_{\alpha'} \circ R_n \circ \Lambda_\alpha$, where $\alpha, \alpha' \in (-1, 1)$.*

In what follows we shall not distinguish between PS equations with parameters related by transformations

$$R_n \to \Lambda_{\alpha'} \circ R_n \circ \Lambda_\alpha, \qquad \alpha, \alpha' \in (-1, 1). \tag{24}$$

In particular, the space of PS_3 equations considered in this paper and of the corresponding monodromies has real dimension 3.

§ 3. Calculation of monodromy for PS_3 equation

We now show the way to an effective calculation of the representation $\mathbf{T}$ in § 2.2 in terms of the coefficients of the function R_n for $n = 3$. To describe the monodromy more geometrically we shall use a graph equipped with matrices the edges of which connect boundary components of the base space. We draw several simple disjoint oriented curves $D_1, D_2, \dots, D_k$ connecting boundary points (punctures) of $\mathcal{Y}$ and equip them with matrices $\mathbf{T}(D_1), \mathbf{T}(D_2), \dots, \mathbf{T}(D_k)$ from $GL_n(\mathbb{C})$. By definition we set the monodromy of a path $\mathbf{r}$ intersecting some cuts $D_{i_1}, D_{i_2}, \dots, D_{i_s}$ transversally one after another to be

$$\mathbf{T}([\mathbf{r}]) := \mathbf{T}^{\varepsilon_1}(D_{i_1})\mathbf{T}^{\varepsilon_2}(D_{i_2}) \cdots \mathbf{T}^{\varepsilon_s}(D_{i_s}),$$

where $\varepsilon_k = 1$ if $\mathbf{r}$ transverses (locally) the cut D_{i_k} from right to left and $\varepsilon_k = -1$ if it transverses the cut in the reverse direction.

We now introduce our notation for 3×3 permutation matrices:

$$\begin{aligned}
\mathbf{D}_1 \cdot (x_1, x_2, x_3)^t &:= (x_1, x_3, x_2)^t, \\
\mathbf{D}_2 \cdot (x_1, x_2, x_3)^t &:= (x_3, x_2, x_1)^t, \\
\mathbf{D}_3 \cdot (x_1, x_2, x_3)^t &:= (x_2, x_1, x_3)^t, \\
\mathbf{D}_0 \cdot (x_1, x_2, x_3)^t &:= (x_3, x_1, x_2)^t.
\end{aligned} \tag{25}$$

3.1. Typical covers. By the Riemann–Hurwitz formula the cover R_3 has in the general case four simple branch points a_1, a_2, a_3, and a_4, so that lying over each point a_k are a double point b_k and a simple point c_k, $k = 1, \dots, 4$. Finding these 12 points requires a solution of an equation of degree 4. Below we classify the functions R_3 in accordance with the position of the branch points with respect to the real axis. This classification will show that one can always connect the branch points by two disjoint cuts $D_1 = a_1\smile a_3$ and $D_2 = a_2\smile a_4$ such that the monodromy of the cover R_3 is trivial over the doubly connected domain $\mathcal{O} := \mathbb{CP}_1 \setminus \{D_1 \cup D_2\}$. We can assume without loss of generality that the cut D lies in $\mathcal{O}$; incidentally, each of its end-points can be a branch point of R_3 (see Fig. 4(a)).

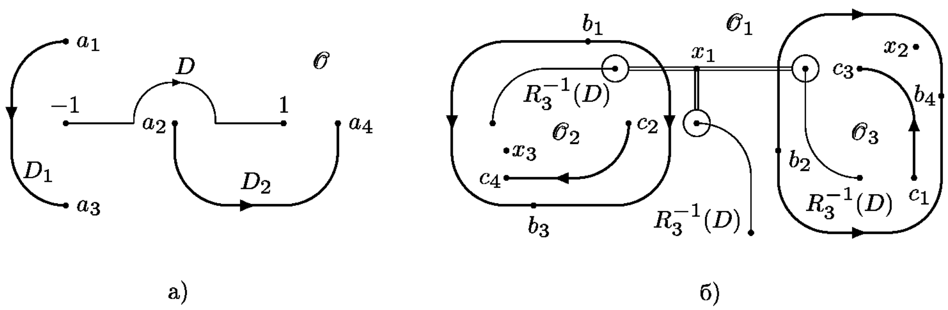


Figure 4. (a) Cuts on $\mathcal{Y}$; (b) attachment of components $\mathcal{O}_k$ and three lassos

Simple combinatorial arguments show that the three components of $R_3^{-1}(\mathcal{O})$ are a (topological) annulus $\mathcal{O}_1$ and two (topological) discs $\mathcal{O}_2$ and $\mathcal{O}_3$ with slits glued as in Fig. 4(b).

Assume that the base point y_0 of the fundamental group $\pi_1(\mathcal{Y})$ lies in $\mathcal{O}$. We number the points lying over it so that $x_1 \in \mathcal{O}_1$, $x_2 \in \mathcal{O}_3$, and $x_3 \in \mathcal{O}_2$. A loop disjoint from the cuts D_1, D_2, and D has the trivial monodromy (see Fig. 4(b)). Loops intersecting once one of the cuts D_k, $k = 1, 2$, have monodromy $\mathbf{D}_{k+1}$ (their orientation at the intersection point is inessential because $\mathbf{D}_{k+1}^2 = \mathbf{1}$). Finally, the monodromy of a loop intersecting the cut D once depends on the component of $R_3^{-1}(\mathcal{O})$ containing $R_3^{-1}(D) \cap \mathcal{U}$, a small deformation of the interval I. Taking for $\mathbf{d}$ the projection of one of the lassos in Fig. 4(b) we see that this monodromy is $\mathbf{D}_k\mathbf{D}\mathbf{D}_k$ if the corresponding component is $\mathcal{O}_k$, $k = 1, 2, 3$. This proves the following result.

Lemma 6. *For a generic function R_3 the monodromy $\mathbf{T}$ is determined up to conjugation by the disjoint cuts $D_1 = a_1\smile a_3$, $D_2 = a_2\smile a_4$, and D and the matrices $\mathbf{D}_2$, $\mathbf{D}_3$, and $\mathbf{D}_k\mathbf{D}\mathbf{D}_k$ assigned to them, where k is such that $\mathcal{O}_k \supset \{R_3^{-1}(D) \cap \mathcal{U}\}$.*

In the remaining part of this section we partition the space of PS$_3$ equations into distinctive blocks and solve two problems in each of them: we determine the configuration of the cuts D, D_1, D_2 and the component $\mathcal{O}_k$ containing (the deformation of) I, which gives us the monodromy $\mathbf{T}$.

3.1.1. Branch points a_1, a_2, a_3, a_4 are real. Note that the points on the extended real axis $\widehat{\mathbb{R}}$ that are distinct from a_1, a_2, a_3, a_4 can be of two *types*: the points y of type 3:0 have three points in the inverse image $R_3^{-1}(y)$, while the inverse images of points of type 1:2 contain one real point and a pair of complex conjugate ones. The type of a point is locally constant and changes on passing over a branch point. In our case there exist precisely two subintervals of $\widehat{\mathbb{R}}$ consisting of points of type 1:2; we denote them by $D_1 := (a_1, a_3)$ and $D_2 := (a_2, a_4)$ (see Fig. 5(a)). The monodromy of the cover is trivial in the complement of these intervals. To prove that $\mathbf{T}_*$ is trivial in $\mathbb{CP}_1 \setminus \{D_1 \cup D_2\}$ we use the following result, which constrains the monodromy of the 'real cover' R_3.

Lemma 7. *Let y_0 be a real base point. If one associates with it the 3×3 permutation matrix $\mathbf{K}$ such that*

$$\mathbf{K}\cdot(x_1, x_2, x_3)^t := (\overline{x}_1, \overline{x}_2, \overline{x}_3)^t,$$

then the following relation holds for the monodromies of complex conjugate loops $\mathbf{r}$ and $\overline{\mathbf{r}}$:

$$\mathbf{T}_*([\overline{\mathbf{r}}]) = \mathbf{K}\mathbf{T}_*([\mathbf{r}])\mathbf{K}. \tag{26}$$

Proof. We have the chain of equalities

$$\begin{aligned}\mathbf{T}_*([\overline{\mathbf{r}}])(x_1, x_2, x_3)^t &:= (x_1\cdot\overline{\mathbf{r}}, x_2\cdot\overline{\mathbf{r}}, x_3\cdot\overline{\mathbf{r}})^t = \overline{(\overline{x}_1\cdot\mathbf{r}, \overline{x}_2\cdot\mathbf{r}, \overline{x}_3\cdot\mathbf{r})}^t \\ &= \mathbf{K}\overline{(x_1\cdot\mathbf{r}, x_2\cdot\mathbf{r}, x_3\cdot\mathbf{r})}^t = \mathbf{K}\mathbf{T}_*([\mathbf{r}])\mathbf{K}(x_1, x_2, x_3)^t.\end{aligned}$$

Consider a loop $\mathbf{r}$ with initial point y_0 of type 3:0 that is symmetric relative to $\widehat{\mathbb{R}}$ and goes around the interval (a_1, a_3) (see Fig. 5(a)). This loop encircles two simple branch points a_1 and a_3, therefore the permutation $\mathbf{T}_*([\mathbf{r}])$ is a product of two transpositions, that is, it is either the identity or a cyclic permutation of order 3. The second case is impossible by Lemma 7, which asserts that $\mathbf{T}_*([\mathbf{r}^{-1}]) = \mathbf{T}_*([\overline{\mathbf{r}}]) \overset{(26)}{=} \mathbf{T}_*([\mathbf{r}])$.

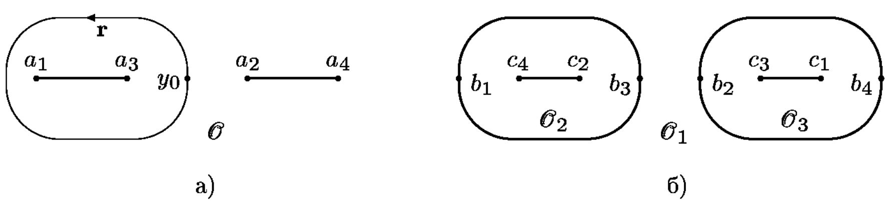


Figure 5. All branch points, a_1, a_2, a_3, and a_4, are real

To find the monodromy $\mathbf{T}$ it suffices by Lemma 6 to find the intersection sets of the extended real axis $\widehat{\mathbb{R}}$ and the components $\overline{\mathcal{O}}_k$, $k = 1, 2, 3$. The extended real axis in the covering sphere is partitioned into eight intervals by the eight points b_k, c_k, $k = 1, \dots, 4$, lying over the branch points. Two of these intervals cover the cuts (a_1, a_3) and (a_2, a_4) and the remaining six cover the complement of these cuts, which consists of the points of type 3:0. The adjacency relations between these eight intervals are uniquely recovered on the basis of the following observation. Two intervals separated by a point b_k cover the same interval with end-point a_k and consisting of 3:0 points. Two intervals separated by c_k cover the two distinct intervals with end-point a_k. In particular, the eight points in the inverse images of the branch points lie on $\widehat{\mathbb{R}}$ in the following order up to direction (see Fig. 5(b)): $b_1, c_4, c_2, b_3, b_2, c_3, c_1, b_4$. Thus, we obtain four well-defined disjoint intervals (b_1, b_3), (b_3, b_2), (b_2, b_4), and (b_4, b_1), which define the required partitioning of the extended real axis of the covering sphere:

$$\overline{\mathcal{O}}_1 \cap \widehat{\mathbb{R}} = [b_3, b_2] \cup [b_4, b_1], \qquad \overline{\mathcal{O}}_2 \cap \widehat{\mathbb{R}} = [b_1, b_3], \qquad \overline{\mathcal{O}}_3 \cap \widehat{\mathbb{R}} = [b_2, b_4].$$

All these arguments prove the following result.

Theorem 2.1.1. *In the case of four real branch points a_1, a_2, a_3, a_4 the monodromy* $\mathbf{T}$ *is defined by the three intervals $D_1 := (a_1, a_3)$, $D_2 := (a_2, a_4)$, and $D := (-1, 1)$ equipped with matrices in accordance with the following table.*

Condition	$\mathbf{T}(D_1)$	$\mathbf{T}(D_2)$	$\mathbf{T}(D)$	Comment
$I \subset (b_1, b_3)$	$\mathbf{D}_2$	$\mathbf{D}_1$	$\mathbf{D}$	+
$I \subset (b_2, b_4)$	$\mathbf{D}_1$	$\mathbf{D}_3$	$\mathbf{D}$	+
$I \subset (b_3, b_2) \cup (b_4, b_1)$	$\mathbf{D}_3$	$\mathbf{D}_2$	$\mathbf{D}$	

Remark. Two intervals, D and D_k, $k = 1, 2$, corresponding to a line in the table marked by '+' may intersect. In that case one must slightly deform them — no matter how — to avoid the intersection. Another solution is to replace the two intersecting intervals by three: their intersection and the two intervals making up the symmetric difference. One assigns to the intersection the product of matrices associated with the intersecting intervals (taken in an arbitrary order because these matrices are $\mathbf{D}_1$ and $\mathbf{D}$), and to the remaining parts the matrices corresponding to the full intervals.

3.1.2. Points a_1, a_3 are real and a_2, a_4 are complex conjugate. If there exist two branch points in the extended real axis, then they are end-points of an interval of 1:2-points. We take this interval for $D_1 := (a_1, a_3)$. For the second cut $D_2 := \smile{a_2 a_4}$ we take an arc that is symmetric relative to $\widehat{\mathbb{R}}$, avoids I, and connects the complex conjugate branch points. In the general case there exist two non-equivalent ways to select such an arc. Changing our notation for the end-points of the first interval if necessary we can assume that D_2 intersects the real axis between a_1 and I (see Fig. 6(a)).

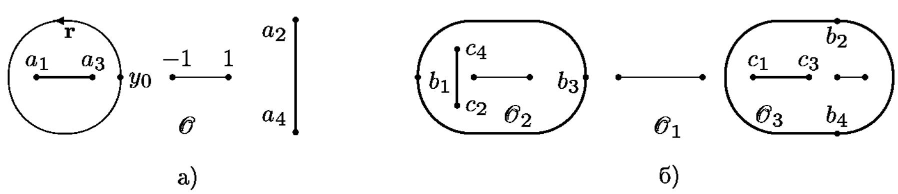


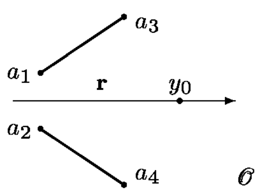
Figure 6. Points a_1, a_3 are real and a_2, a_4 are complex conjugate

Repeating word for word the paragraph following Lemma 7 we can prove that the monodromy of the cover is trivial over the resulting doubly connected domain $\mathcal{O}$. Lifting the interval (a_1, a_3) and its complement (a_3, a_1), which consists of 3:0-points, to the covering sphere we see that the inverse images of the real branch points a_1 and a_3 lie on $\widehat{\mathbb{R}}$ in the following order (up to direction): b_1, b_3, c_1, c_3. Thus, we have the four well-defined disjoint intervals (b_1, b_3), (b_3, c_1), (c_1, c_3), and (c_3, b_1) of the extended real axis. We use this notation in the statement of the following result (see Fig. 6(b)).

Theorem 2.1.2. *In the case of two real branch points a_1 and a_3 and two complex conjugate ones, a_2 and a_4, the monodromy $\mathbf{T}$ is defined by the three cuts $D_1 = (a_1, a_3)$, $D_2 = \overset{\smile}{a_2 a_4}$, and $D := (-1, 1)$ equipped with matrices in accordance with the following table.*

Condition	$\mathbf{T}(D_1)$	$\mathbf{T}(D_2)$	$\mathbf{T}(D)$	Comment
$I \subset (b_1, b_3)$	$\mathbf{D}_2$	$\mathbf{D}_1$	$\mathbf{D}$	
$I \cap (c_1, c_3) \neq \varnothing$	$\mathbf{D}_1$	$\mathbf{D}_3$	$\mathbf{D}$	+
$I \subset (c_3, b_1)$	$\mathbf{D}_1$	$\mathbf{D}_3$	$\mathbf{D}$	
$I \subset (b_3, c_1)$	$\mathbf{D}_3$	$\mathbf{D}_2$	$\mathbf{D}$	

3.1.3. Two pairs of conjugate branch points; $\widehat{\mathbb{R}}$ has type 1:2. In the absence of branch points on the real axis all its points have the same type, which is 1:2 in the present case. We take for the cuts the two complex conjugate straight line intervals $D_1 := (a_1, a_3)$ and $D_2 := (a_2, a_4)$ lying entirely in the upper and lower half-planes, respectively (see Fig. 7(a)).



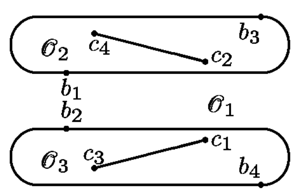


Figure 7. (a) Four complex branch points; (b) $R_3^{-1}(\widehat{\mathbb{R}})$ consists of three components

Let $\mathbf{r}$ be the real axis. As before, $\mathbf{T}_*([\mathbf{r}])$ is the identity or a cyclic permutation of order 3. The latter is ruled out by Lemma 7, which asserts that we have $\mathbf{T}_*([\mathbf{r}]) = \mathbf{T}_*([\overline{\mathbf{r}}]) \overset{(26)}{=} \mathbf{K}\mathbf{T}_*([\mathbf{r}])\mathbf{K} = \mathbf{T}_*^{-1}([\mathbf{r}])$, where $\mathbf{K}$ is a second-order permutation. For the configuration in question we always have $I \subset \mathcal{O}_1$ (see Fig. 7(b)), therefore the following result holds.

Theorem 2.1.3. *If the cover R_3 has two pairs of conjugate branch points $a_1 = \overline{a}_2$ and $a_3 = \overline{a}_4$ and the real axis consists of points of type* 1:2, *then the monodromy $\mathbf{T}$ is defined by the three disjoint straight line cuts $D_1 := (a_1, a_3)$, $D_2 := (a_2, a_4)$, and $D := (-1, 1)$ equipped with the matrices $\mathbf{D}_3$, $\mathbf{D}_2$, and $\mathbf{D}$, respectively.*

3.1.4. Two pairs of complex conjugate branch points; $\widehat{\mathbb{R}}$ has type 3:0. In the remaining case all points in the extended real axis have type 3:0. Joining the complex conjugate branch points by cuts $D_1 := \overset{\smile}{a_1 a_3}$ and $D_2 := \overset{\smile}{a_2 a_4}$ avoiding I as in Fig. 8(a) and symmetric relative to $\widehat{\mathbb{R}}$ we obtain a doubly connected domain $\mathcal{O}$ with trivial monodromy of the cover R_3. For the proof consider a symmetric loop $\mathbf{r}$ with homotopy class generating $\pi_1(\mathcal{O})$. This loop encircles a pair of branch points of R_3 and $\overline{\mathbf{r}} \sim \mathbf{r}^{-1}$, so that the monodromy along $\mathbf{r}$ is trivial (see the paragraph following the proof of Lemma 7).

We now find the partitioning of the extended real axis by the components $\overline{\mathcal{O}}_j$. We denote the points in the inverse image of the intersection of D_k, $k = 1, 2$, and $\widehat{\mathbb{R}}$ by $\{e_1^k, e_2^k, e_3^k\}$, where e_1^k lies on the component of the inverse image of the cut that connects c_k and c_{k+2}. Lifting the extended real axis to the covering sphere one interval after another we see that the points e_*^* with superscripts 1 and 2 alternate, while the interval between successive points with subscript 1 contains two points with other subscripts. We shall assume without loss of generality that these points lie on the extended axis in the following order: $e_1^1, e_2^2, e_3^1, e_1^2, e_2^1, e_3^2$, so that we have four well-defined disjoint intervals (e_3^1, e_2^1), (e_2^1, e_3^2), (e_3^2, e_2^2), and (e_2^2, e_3^1). These intervals make up the required partitioning (see Fig. 8(b)):

$$\overline{\mathcal{O}}_1 \cap \widehat{\mathbb{R}} = [e_2^1, e_3^2] \cup [e_2^2, e_3^1], \qquad \overline{\mathcal{O}}_2 \cap \widehat{\mathbb{R}} = [e_3^1, e_2^1], \qquad \overline{\mathcal{O}}_3 \cap \widehat{\mathbb{R}} = [e_3^2, e_2^2].$$

All these arguments prove the following result.

Theorem 2.1.4. *If the cover R_3 has two pairs of complex conjugate branch points $a_1 = \overline{a}_3$ and $a_2 = \overline{a}_4$ and the real axis consists of points of type 3:0, then the monodromy is defined by the three disjoint cuts $D_1 := \overset{\smile}{a_1 a_3}$, $D_2 := \overset{\smile}{a_2 a_4}$, and $D := (-1, 1)$, that are symmetric with respect to $\widehat{\mathbb{R}}$ and are equipped with matrices in accordance with the following table.*

Condition	$\mathbf{T}(D_1)$	$\mathbf{T}(D_2)$	$\mathbf{T}(D)$
$I \subset (e_3^1, e_2^1)$	$\mathbf{D}_2$	$\mathbf{D}_1$	$\mathbf{D}$
$I \subset (e_3^2, e_2^2)$	$\mathbf{D}_1$	$\mathbf{D}_3$	$\mathbf{D}$
$I \subset (e_2^1, e_3^2) \cup (e_2^2, e_3^1)$	$\mathbf{D}_3$	$\mathbf{D}_2$	$\mathbf{D}$

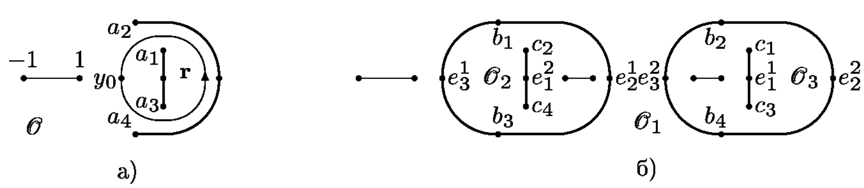


Figure 8. (a) Four complex branch points; (b) $R_3^{-1}(\widehat{\mathbb{R}})$ consists of one component

3.2. Multiplicity-one degeneracy of the cover. By a multiplicity-one degeneracy we mean the coalescence of two simple branch points of a cover into one 'complex' branch point. All results in this section can be obtained by formally setting $a_1 = a_4$ in § 3.1.

Let a_2 and a_3 be simple branch points of R_3 and let a_1 be a 'complex' branch point, that is, there is a triple point b_1 over it. We connect the three branch points by simple disjoint arcs $D_1 := \overset{\smile}{a_1 a_3}$ and $D_2 := \overset{\smile}{a_2 a_1}$ avoiding the cut D (see Fig. 9(a)). The three components of the inverse image of the simply connected domain $\mathcal{O} := \mathbb{CP}_1 \setminus \{D_1 \cup D_2\}$ are glued as in Fig. 9(b). The following result is a complete analogue of Lemma 6.

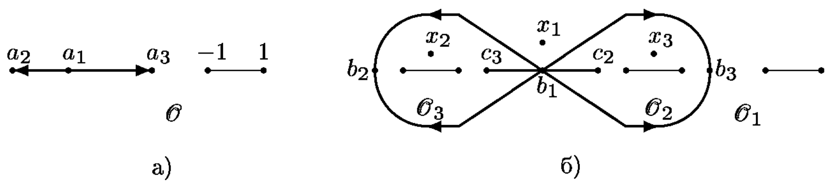


Figure 9. Simple degeneracy of the cover R_3: (a) cuts of the base $\mathcal{Y}$; (b) gluing the components $\mathcal{O}_k$

Lemma 8. *Up to conjugation, in the case of a multiplicity-one degeneracy of the cover R_3 the monodromy $\mathbf{T}$ is defined by the three disjoint cuts $D_1 := \overset{\smile}{a_1 a_3}$, $D_2 := \overset{\smile}{a_2 a_1}$, and D equipped with the matrices $\mathbf{D}_2$, $\mathbf{D}_3$, and $\mathbf{D}_k\mathbf{D}\mathbf{D}_k$, respectively, where k is such that $\mathcal{O}_k \supset \{R_3^{-1}(D) \cap \mathcal{U}\}$.*

The blocks in the space of PS_3 equations defined below have real dimension 2; they lie at the boundary of the three-dimensional blocks considered in § 3.1.

3.2.1. The simple branch points a_2 and a_3 are real. This is the limit case of § 3.1.1. The extended real axis of the base sphere contains two intervals with end-point a_1 consisting of 1:2-points. We set these intervals to be the cuts (a_1, a_3) and (a_2, a_1). The points over the branch points lie on the 'circle' $\widehat{\mathbb{R}}$ in the following order: b_1, c_2, b_3, b_2, c_3. They define a partitioning of this circle by the components $\overline{\mathcal{O}}_k$:

$$\overline{\mathcal{O}}_1 \cap \widehat{\mathbb{R}} = [b_3, b_2], \qquad \overline{\mathcal{O}}_2 \cap \widehat{\mathbb{R}} = [b_1, b_3], \qquad \overline{\mathcal{O}}_3 \cap \widehat{\mathbb{R}} = [b_2, b_1].$$

The closed intervals here are the closures of the disjoint open intervals (b_1, b_2), (b_2, b_3), and (b_3, b_1) from the partitioning of the extended real axis by the points b_1, b_2, b_3. This proves the following result.

Theorem 2.2.1. *In the case of simple real branch points a_2 and a_3 the monodromy $\mathbf{T}$ is defined by the three intervals $D_1 := (a_1, a_3)$, $D_2 := (a_2, a_1)$, and $D := (-1, 1)$ equipped with matrices in accordance with the following table.*

Condition	$\mathbf{T}(D_1)$	$\mathbf{T}(D_2)$	$\mathbf{T}(D)$	Comment
$I \subset (b_1, b_3)$	$\mathbf{D}_2$	$\mathbf{D}_1$	$\mathbf{D}$	$+$
$I \subset (b_2, b_1)$	$\mathbf{D}_1$	$\mathbf{D}_3$	$\mathbf{D}$	$+$
$I \subset (b_2, b_3)$	$\mathbf{D}_3$	$\mathbf{D}_2$	$\mathbf{D}$	

3.2.2. The simple branch points a_2 and a_3 are complex conjugate. This is the limit case of the configuration in § 3.1.3. We join a_1 to a_2 and a_3 by complex conjugate straight line cuts. The cut I of the covering Riemann sphere lies in the component $\overline{\mathcal{O}}_1$, therefore we have the following result.

Theorem 2.2.3. *If the cover R_3 has a pair of complex conjugate simple branch points $a_2 = \overline{a}_3$, then the monodromy $\mathbf{T}$ is defined by the three disjoint straight-line cuts $D_1 := (a_1, a_3)$, $D_2 := (a_2, a_1)$, and $D := (-1, 1)$ equipped with the matrices $\mathbf{D}_3$, $\mathbf{D}_2$, and $\mathbf{D}$, respectively.*

3.3. Multiplicity-two degeneration of the cover. Consider now the case of two 'complex' branch points a_1 and a_2, real or complex conjugate. We connect these branch points by a simple arc $D_0 := \smile{a_1 a_2}$ avoiding the cut I. We have the following result.

Theorem 2.3. *If R_3 is a cover with two branch points a_1 and a_2, then the monodromy $\mathbf{T}$ is defined up to conjugation by the two disjoint cuts $D_0 := \smile{a_1 a_2}$ and $D := (-1, 1)$ equipped with the matrices $\mathbf{D}_0$ and $\mathbf{D}$, respectively.*

The change of the orientation of D_0 corresponds to the conjugation of the representation $\mathbf{T}$ by the matrix $\mathbf{D}_1$, therefore we can fix an arbitrary orientation on $\smile{a_1 a_2}$.

§ 4. Monodromy problem on a Riemann surface

Each puncture $a = \pm 1, a_1, a_2, a_3, a_4$ of $\mathcal{Y}$ is a branch point of the solution $W(y)$ of the Riemann monodromy problem. The branching order $r(a)$ is finite and equal to 2 or 3, for the local monodromy matrix is similar to one of the matrices $\mathbf{D}$, $\mathbf{D}_0$, $\mathbf{D}_1$, $\mathbf{D}_2$, $\mathbf{D}_3$, and $\mathbf{D}\mathbf{D}_1 = \mathbf{D}_1\mathbf{D}$. It will be convenient to proceed to a monodromy problem such that its solution has no local branchings. To this end we introduce the compact Riemann surface $\mathcal{M} = \mathcal{M}(R_3)$ with ramification of order $r(a)$ over each puncture a. For instance, in the general position case all the six punctures a are distinct, all the orders $r(a)$ are equal to 2, and therefore $\mathcal{M} = \{w^2 = (y^2 - 1)\prod_{k=1}^4 (y - a_k)\}$ is a hyperelliptic surface of genus 2. Lifted to $\mathcal{M}$, the solution $W(y)$ of the monodromy problem in Theorem 1 becomes a holomorphic vector $W_{\mathcal{M}}$ with new monodromy $\mathbf{T}_{\mathcal{M}}$ and with additional symmetries connected with conformal motions of $\mathcal{M}$.

4.1. The Riemann surface $\mathcal{M}$. Let $D_1 = \smile{a_1 a_3}$, $D_2 = \smile{a_2 a_4}$, $D = \smile{-1 1}$ be the cuts of $\mathcal{Y}$ defining the monodromy of $\mathbf{T}$ (in the case of double degeneracy of R_3 we consider the cuts D and D_0). We define a representation $\mathbf{T}_{*l}$ of the fundamental group $\mathcal{Y}$ into the symmetric group S_l by assigning to each cut a permutation so as to take account of the number of punctures a and the corresponding orders $r(a)$, in accordance with Table 1.

The expression $\mathbf{2}_\alpha \mathbf{3}_\beta$ in the first column of the table means that $\mathcal{Y}$ has α punctures with $r = 2$ and β punctures with $r = 3$. The second column indicates the order of the symmetric group. In the columns from third to sixth we list the permutations corresponding to the cuts of the punctured sphere $\mathcal{Y}$.

We consider now l copies of the surface $\mathcal{Y} \setminus \{D \cup D_1 \cup D_2\}$ (or — in the case of a multiplicity-two degeneracy — of $\mathcal{Y} \setminus \{D \cup D_0\}$) and glue the right bank of each cut $D_\bullet$ on the kth sheet with the left bank of a similar cut on the sth sheet in the

Table 1

R_3	l	$\mathbf{T}_{*l}(D_\bullet)(m_1, m_2, \dots, m_l)$				$g(\mathcal{M})$	$\dim H^1_s(\mathcal{M})$
		$D_\bullet = D_1$	$D_\bullet = D_2$	$D_\bullet = D$	$D_\bullet = D_0$		
$\mathbf{2}_6\mathbf{3}_0$	2	(m_2, m_1)	(m_2, m_1)	(m_2, m_1)	–	2	2
$\mathbf{2}_5\mathbf{3}_0$	4	(m_4, m_3, m_2, m_1)	(m_4, m_3, m_2, m_1)	(m_2, m_1, m_4, m_3)	–	2	1
$\mathbf{2}_4\mathbf{3}_0$	4	(m_2, m_1, m_4, m_3)	(m_4, m_3, m_2, m_1)	(m_1, m_2, m_3, m_4)	–	1	0
$\mathbf{2}_4\mathbf{3}_1$	6	$(m_2, m_1, m_4, m_3, m_6, m_5)$	$(m_6, m_3, m_2, m_5, m_4, m_1)$	$(m_6, m_3, m_2, m_5, m_4, m_1)$	–	3	1
$\mathbf{2}_3\mathbf{3}_1$	12	$(m_2, m_1, m_4, m_3, m_6, m_5, m_8, m_7, m_{10}, m_9, m_{12}, m_{11})$	$(m_6, m_3, m_2, m_5, m_4, m_1, m_{12}, m_9, m_8, m_{11}, m_{10}, m_7)$	$(m_7, m_8, m_9, m_{10}, m_{11}, m_{12}, m_1, m_2, m_3, m_4, m_5, m_6)$	–	2	0
$\mathbf{2}_2\mathbf{3}_2$	6	–	–	$(m_4, m_5, m_6, m_1, m_2, m_3)$	$(m_2, m_3, m_1, m_5, m_6, m_4)$	2	0

case when $\mathbf{T}_{*l}$ takes m_k to m_s, $k = 1, \dots, l$; here $D_\bullet = D, D_1, D_2$. We denote the resulting surface by $\mathcal{M}_\circ$, seal the punctures inherited from $\mathcal{Y}$, and obtain a compact surface $\mathcal{M}$. We depict two such surfaces in Fig. 10(a),(b). If m_k is a point lying over y_0 on the kth sheet, then $\mathbf{T}_{*l}$ is the monodromy of the (regular, as we show below) l-sheeted cover pr in the diagram

$$\begin{array}{ccc} \mathcal{M}_\circ & \xrightarrow{\imath} & \mathcal{M} \\ \Big\downarrow{\scriptstyle pr} & & \Big\downarrow{\scriptstyle pr} \\ \mathcal{Y} & \xrightarrow{\imath} & \mathbb{CP}_1 \end{array} . \tag{27}$$

Here $\imath$ is the natural embedding.

Constructed as an l-sheeted cover of the Riemann sphere, $\mathcal{M}$ is not the unique surface suitable for our aims, but it is in a certain sense the simplest one: for instance, it has the smallest genus $g(\mathcal{M})$.

4.2. Geometry of surface $\mathcal{M}$. In the study of the monodromy we require information on the geometry of the surface $\mathcal{M}$ constructed in § 4.1 and, in particular, on the group of its conformal motions and the existence of holomorphic differentials with a certain symmetry.

We have the following result.

Lemma 9. (i) *The cover pr in* (27) *is ramified of order $r(a)$ over each puncture a of the surface $\mathcal{Y}$.*

(ii) *The genus of $\mathcal{M}$ is as indicated in the seventh column of Table* 1.

(iii) *Let pr_* and $\imath_*$ be the maps of the fundamental groups induced by the diagram* (27); *then the subgroups $pr_*\pi_1(\mathcal{M}_\circ, m_1)$ and $pr_* \ker \imath_*$ are normal in $\pi_1(\mathcal{Y}, y_0)$. The first subgroup is the kernel of the representation $\mathbf{T}_{*l}$, while the second lies in the kernel of $\mathbf{T}$, which enables one to define in a natural way the monodromy $\mathbf{T}_\mathcal{M}$ making the following diagram commutative*:

$$\begin{tikzcd}
\pi_1(\mathcal{M}_\circ, m_1) \arrow[r, "\imath_*"] \arrow[d, "pr_*"] & \pi_1(\mathcal{M}, m_1) \arrow[d, "\mathbf{T}_\mathcal{M}"] \\
\pi_1(\mathcal{Y}, y_0) \arrow[r, "\mathbf{T}"] & GL_3(\mathbb{C})
\end{tikzcd} \quad . \tag{28}$$

Proof. (i) We verify the lemma, for instance, in the case $\mathbf{2}_4\mathbf{3}_1$ corresponding to the polynomial R_3 in general position. In this case there exist four punctures a with $r(a) = 2$ (the free end-points of the cuts D, D_1, and D_2) and one puncture with $r(a) = 3$ (the common end-point of D_1 and D_2 — see Fig. 11). Going around the first four points corresponds to the permutations $\mathbf{T}_{*6}(D)$, $\mathbf{T}_{*6}(D_1)$, and $\mathbf{T}_{*6}(D_2)$, which are formed by cycles of length 2; the fifth point corresponds to the permutation $\mathbf{T}_{*6}(D_1)\mathbf{T}_{*6}(D_2)$, formed by cycles of length 3.

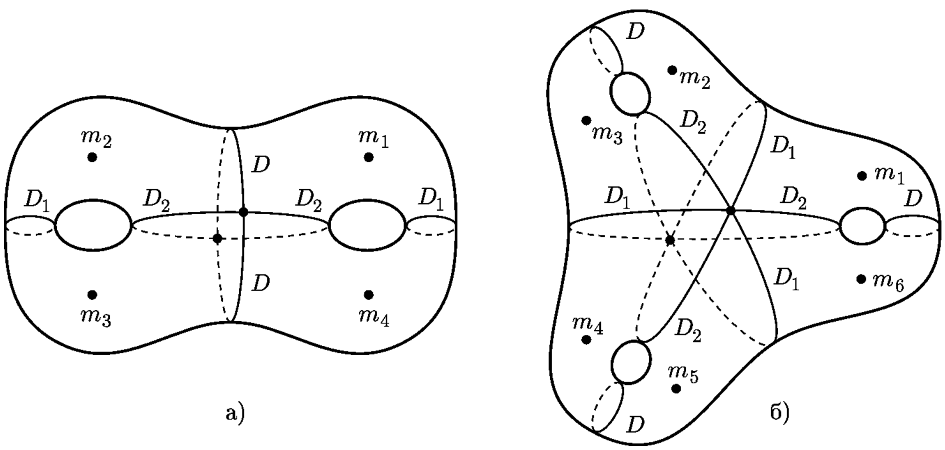


Figure 10. Surface $\mathcal{M}$ in cases (a) $\mathbf{2}_5\mathbf{3}_0$; (b) $\mathbf{2}_4\mathbf{3}_1$

(ii) We apply the Riemann–Hurwitz formula to the cover pr.

(iii) We claim that the covering group of pr is $\ker \mathbf{T}_{*l}$ and it is therefore a normal subgroup of $\pi_1(\mathcal{Y})$. Let m_1 be the base point of the fundamental group of $\mathcal{M}_\circ$. Then the projection of $\pi_1(\mathcal{M}_\circ, m_1)$ onto $\pi_1(\mathcal{Y}, y_0)$ is generated by the classes of loops $\mathbf{r} \subset \mathcal{Y}$ such that the permutation $\mathbf{T}_{*l}([\mathbf{r}])$ fixes the first element. As regards the monodromy group, the permutations in it, except for the trivial one, have no cycles of length 1. We demonstrate this in case $\mathbf{2}_4\mathbf{3}_1$. We consider two permutations commuting with the permutations in the monodromy group $\mathbf{T}_{*6}$:

$$\mathbf{S}_1(m_1, m_2, \dots, m_6) = (m_3, m_4, m_5, m_6, m_1, m_2),$$
$$\mathbf{S}_2(m_1, m_2, \dots, m_6) = (m_2, m_1, m_6, m_5, m_4, m_3).$$

If $\mathbf{K} := \mathbf{T}_{*6}([\mathbf{r}])$ fixes the first element, then the permutations $\mathbf{S}_1^{\pm 1}\mathbf{K}\mathbf{S}_1^{\mp 1}$ $(= \mathbf{K})$ fix the third and the fifth elements and the permutation $\mathbf{S}_2\mathbf{K}\mathbf{S}_2^{-1}$ $(= \mathbf{K})$ fixes the second, fourth, and sixth elements. Hence the permutation $\mathbf{K}$ is trivial.

We proceed now to the second subgroup $pr_* \ker \imath_*$. The kernel of $\imath_*$ is generated by the lassos in $\mathcal{M}_\circ$ encircling the punctures. Accordingly, the subgroup in question is generated by the lassos in $\mathcal{Y}$ that encircle the punctures a and make $r(a)$ circuits about them. This class of generators is invariant under conjugations in $\pi_1(\mathcal{Y}, y_0)$ and by part (i) of the lemma lies in the kernel of $\mathbf{T}$.

4.2.1. Motions of the surface. The fundamental group $\pi_1(\mathcal{Y})$ acts in the natural way in $\mathcal{M}$ and its universal cover $\widetilde{\mathcal{M}}$, so that $\mathcal{M}/\pi_1(\mathcal{Y}) = \widetilde{\mathcal{M}}/\pi_1(\mathcal{Y}) = \mathbb{CP}_1$. For a point $m \in \mathcal{M}_\circ$ this action is described by the formula

$$[\mathbf{r}] \cdot m = [\mathbf{r}] \cdot (m_1 \cdot \mathbf{s}) := m_1 \cdot \mathbf{rs}, \qquad [\mathbf{r}] \in \pi_1(\mathcal{Y})/pr_*\pi_1(\mathcal{M}_\circ),$$

where $\mathbf{s}$ is the projection onto $\mathcal{Y}$ of a path in $\mathcal{M}_\circ$ joining the base point m_1 to the variable point m. Since $pr_*\pi_1(\mathcal{M}_\circ)$ is a normal subgroup of $\pi_1(\mathcal{Y})$, the right-hand side of the formula is independent of one's choice of the path $\mathbf{s}$.

The definition of the action of $\pi_1(\mathcal{Y})$ in the universal cover $\widetilde{\mathcal{M}}$ is only slightly more complicated. The embedding $\imath\colon \mathcal{M}_\circ \to \mathcal{M}$ generates a map of universal covers:

$$\begin{array}{ccc} \widetilde{\mathcal{Y}} = \widetilde{\mathcal{M}}_\circ & \xrightarrow{\tilde{\imath}} & \widetilde{\mathcal{M}} \\ \Big\downarrow & & \Big\downarrow \\ \mathcal{M}_\circ & \xrightarrow{\imath} & \mathcal{M} \end{array} \quad . \tag{29}$$

We select in $\widetilde{\mathcal{Y}}$ a base point $\widetilde{y}_0$ lying over m_1, and for the points $\tilde{\imath}(\widetilde{y}_0 \cdot \mathbf{s}) = \widetilde{m} \in \widetilde{\mathcal{M}}$ with projections lying outside the punctures of $\mathcal{M}_\circ$ we set

$$[\mathbf{r}] \cdot \widetilde{m} = [\mathbf{r}] \cdot \tilde{\imath}(\widetilde{y}_0 \cdot \mathbf{s}) := \tilde{\imath}(\widetilde{y}_0 \cdot \mathbf{r} \cdot \mathbf{s}), \qquad \mathbf{s} \subset \mathcal{Y}, \quad [\mathbf{r}] \in \pi_1(\mathcal{Y})/pr_* \ker \imath_*. \tag{30}$$

The result of this action does not depend on our choice of $\mathbf{s}$: if $\tilde{\imath}$ glues together points $\widetilde{y}_0 \cdot \mathbf{s}_1$ and $\widetilde{y}_0 \cdot \mathbf{s}_2$, then $[\mathbf{s}_1 \cdot \mathbf{s}_2^{-1}] \in pr_* \ker \imath_*$. Since $pr_* \ker \imath_*$ is a normal subgroup of $\pi_1(\mathcal{Y})$, the element $[\mathbf{rs}_1 \cdot (\mathbf{rs}_2)^{-1}]$ also lies in $pr_* \ker \imath_*$, and therefore $\tilde{\imath}$ glues together the points $\widetilde{y}_0 \cdot \mathbf{rs}_1$ and $\widetilde{y}_0 \cdot \mathbf{rs}_2$.

The action of $\pi_1(\mathcal{Y})$ on the punctures of $\mathcal{M}_\circ$ and on the points in the universal cover $\widetilde{\mathcal{M}}$ lying over them is defined by continuity.

4.2.2. Even paths and symmetric differentials on $\mathcal{M}$. We start with two definitions. Let $\operatorname{sign}\colon \pi_1(\mathcal{Y}) \to \{\pm 1\}$ be the representation such that

$$\operatorname{sign}([\mathbf{r}]) := \det \mathbf{T}([\mathbf{r}]), \qquad [\mathbf{r}] \in \pi_1(\mathcal{Y}). \tag{31}$$

We say that the classes of loops in the kernel of sign are *even*, while the rest are *odd*. The parity of the class of a loop is equal to the parity of the number of intersections of this loop with the cuts D, D_1, and D_2, provided that these intersections are transversal.

We say that a holomorphic differential ω on $\mathcal{M}$ is *symmetric* if

$$\omega([\mathbf{r}]\cdot m) = \operatorname{sign}([\mathbf{r}])\omega(m), \qquad m \in \mathcal{M}, \quad [\mathbf{r}] \in \pi_1(\mathcal{Y})/pr_*\pi_1(\mathcal{M}_\circ). \tag{32}$$

The sets of even loops and symmetric differentials are described by the two lemmas below.

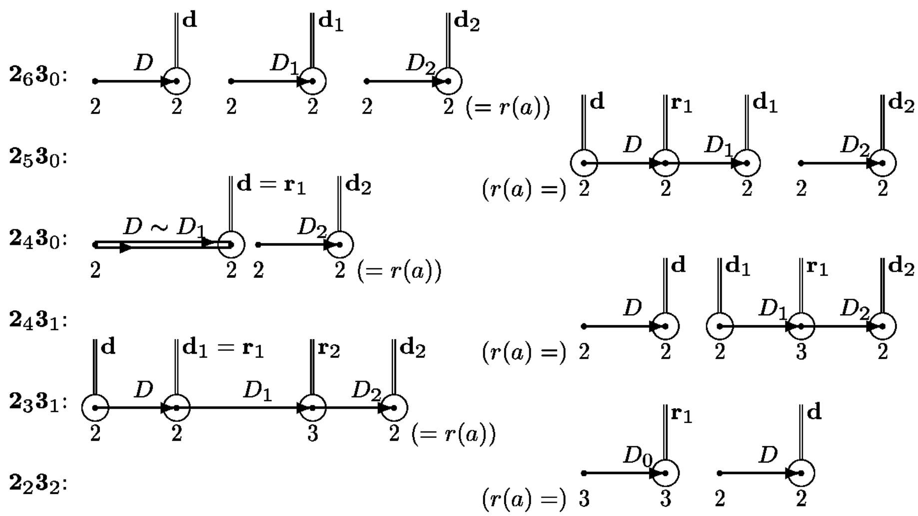


Figure 11. Arrangements of cuts and the generators $\mathbf{d}, \mathbf{d}_1, \mathbf{d}_2; \mathbf{r}_1, \mathbf{r}_2 \in \pi_1(\mathcal{Y}, y_0)$

Lemma 10. *The group* $\ker \operatorname{sign}$ *of even loops on the surface* $\mathcal{Y}$ *is generated by the covering group* $pr_*\pi_1(\mathcal{M}_\circ)$ *and one* (*in cases* $\mathbf{2}_5\mathbf{3}_0$, $\mathbf{2}_4\mathbf{3}_1$, $\mathbf{2}_4\mathbf{3}_0$, *and* $\mathbf{2}_2\mathbf{3}_2$) *or both* (*in case* $\mathbf{2}_3\mathbf{3}_1$) *of the classes* $[\mathbf{r}_1]$ *and* $[\mathbf{r}_2]$ *with representatives indicated in Fig.* 11.

Proof. A direct verification on the generators of the fundamental group of the covering space shows that all loops in $pr_*\pi_1(\mathcal{M}_\circ)$ are even. We now define the missing generators. The classes of the three lassos $\mathbf{d}$, $\mathbf{d}_1$, and $\mathbf{d}_2$ in Fig. 11 (the cases $\mathbf{2}_4\mathbf{3}_0$ and $\mathbf{2}_2\mathbf{3}_2$ require separate consideration, but the corresponding arguments are similar to the ones below) and the subgroup $\pi_1(\mathcal{Y} \setminus \{D \cup D_1 \cup D_2\})$ generate the entire fundamental group $\pi_1(\mathcal{Y}, y_0)$. The fundamental group of the sphere with cuts lies in the covering group of pr, therefore each even element of $\pi_1(\mathcal{Y})$ has a representation $[\mathbf{md}_\star]$ such that $[\mathbf{m}] \in pr_*\pi_1(\mathcal{M}_\circ)$, and $[\mathbf{d}_\star]$ is the product of an even number of elements of the set $\{[\mathbf{d}]^{\pm 1}, [\mathbf{d}_1]^{\pm 1}, [\mathbf{d}_2]^{\pm 1}\}$. The element $[\mathbf{d}_\star]$ can be represented as a product of the squares of the elements $[\mathbf{d}]$, $[\mathbf{d}_1]$, $[\mathbf{d}_2]$, which always belong to $\ker \mathbf{T}_{*l}$, and the pairwise products $[\mathbf{dd}_2]$, $[\mathbf{d}_1\mathbf{d}_2]$. In case $\mathbf{2}_3\mathbf{3}_1$ the last two products do not belong to the kernel of $\mathbf{T}_{*l}$; in cases $\mathbf{2}_5\mathbf{3}_0$ and $\mathbf{2}_4\mathbf{3}_1$ one of these products does not belong to the kernel of $\mathbf{T}_{*l}$; finally, in general-position case $\mathbf{2}_6\mathbf{3}_0$ both products are in $\ker \mathbf{T}_{*l}$. The pairwise products that do not occur in $\ker \mathbf{T}_{*l} = pr_*\pi_1(\mathcal{M}_\circ)$ coincide up to an element of the covering group of pr with the classes $[\mathbf{r}_1]$ and $[\mathbf{r}_2]$ indicated in Fig. 11.

Lemma 11. *The dimension of the space $H^1_s(\mathcal{M})$ of symmetric holomorphic differentials ω on the surface $\mathcal{M}$ is as indicated in the last column of Table* 1.

Proof. Let ω be a symmetric differential on $\mathcal{M}$. Equalities (32) for even motions $[\mathbf{r}]$ show that ω projects onto a holomorphic differential on the quotient of $\mathcal{M}$ by the group $\ker \operatorname{sign}/pr_*\pi_1(\mathcal{M}_\circ)$ of its even motions. However, the surface

$$\mathcal{N} = \mathcal{M}/(\ker \operatorname{sign}/pr_*\pi_1(\mathcal{M}_\circ)),$$

which covers $\mathbb{CP}_1$ in a two-sheeted way, is a Riemann sphere in cases $\mathbf{2}_4\mathbf{3}_0$, $\mathbf{2}_3\mathbf{3}_1$, and $\mathbf{2}_2\mathbf{3}_2$; a torus in cases $\mathbf{2}_5\mathbf{3}_0$ and $\mathbf{2}_4\mathbf{3}_1$; and it is the surface $\mathcal{M}$ itself in general-position case $\mathbf{2}_6\mathbf{3}_0$. There exist no holomorphic differentials on the sphere, only one differential on a torus and a pair of linearly independent holomorphic differentials on a surface of genus 2. Lifting these differentials from $\mathcal{N}$ to $\mathcal{M}$ we always obtain symmetric differentials, because an odd motion of $\mathcal{M}$ results in a permutation of sheets (= an involution) on $\mathcal{N}$ and in a change of sign of a holomorphic differential on $\mathcal{N}$.

If $\mathcal{N}$ is a torus, then the divisor (ω) of a symmetric differential is precisely the ramification divisor of the cover $\mathcal{M} \to \mathcal{N}$, that is, the support of (ω) is the pair of points fixed by the rotation $[\mathbf{r}_1]$ of the surface $\mathcal{M}$; they are labelled by bold dots in Fig. 10(a),(b).

4.3. Lifting the monodromy problem. The solution $W(\widetilde{y})$ of the Riemann problem from Theorem 1 can be lifted to $\mathcal{M}$. The resulting vector $W_{\mathcal{M}}$ is analytic on $\mathcal{M}$ and has monodromy $\mathbf{T}_{\mathcal{M}}$; it also has additional symmetries related to the motions $\mathcal{M}$ in the covering group of the (ramified) cover pr.

Let $\widetilde{m} = \widetilde{\imath}(\widetilde{y}_0 \cdot \mathbf{s})$ be a point in $\widetilde{\mathcal{M}}$ with projection outside the punctures of $\mathcal{M}_\circ$. We set by definition

$$W_{\mathcal{M}}(\widetilde{m}) = W_{\mathcal{M}}(\widetilde{\imath}(\widetilde{y}_0 \cdot \mathbf{s})) := W(\widetilde{y}_0 \cdot \mathbf{s}), \qquad \mathbf{s} \in \mathcal{Y}. \tag{33}$$

This is a consistent definition: if the map $\widetilde{\imath}$ glues together a pair of points $\widetilde{y}_0 \cdot \mathbf{s}_1$ and $\widetilde{y}_0 \cdot \mathbf{s}_2$ in the universal cover $\widetilde{\mathcal{Y}}$, then the class $[\mathbf{s}_1 \cdot \mathbf{s}_2^{-1}]$ lies in $pr_* \ker \imath_*$ and — by Lemma 9(iii) — in the kernel of $\mathbf{T}$, so that the vector W takes equal values at these two points. The set of points in $\widetilde{\mathcal{M}}$ lying over the punctures of $\mathcal{M}_\circ$ is discrete and we can define $W_{\mathcal{M}}$ at these points by continuity because the solution W is bounded in the neighbourhood of the punctures of $\mathcal{Y}$.

The vector $W_{\mathcal{M}}$ defined in this way inherits, of course, the symmetries (20) of W. In terms of the action of the fundamental group $\pi_1(\mathcal{Y})$ in the cover $\widetilde{\mathcal{M}}$ the transformation formula of $W_{\mathcal{M}}$ can be written as follows:

$$W_{\mathcal{M}}([\mathbf{r}] \cdot \widetilde{m}) = \mathbf{T}([\mathbf{r}])W_{\mathcal{M}}(\widetilde{m}), \qquad \widetilde{m} \in \widetilde{M}, \quad [\mathbf{r}] \in \pi_1(\mathcal{Y}, y_0). \tag{34}$$

In particular, the monodromy of $W_{\mathcal{M}}$ on $\mathcal{M}$ is $\mathbf{T}_{\mathcal{M}}$.

All these arguments are reversible, therefore we have the following result.

Theorem 3. *The eigenvalue-eigenvector pairs $(\lambda, u(x))$ of the* PS_3 *equation for $\lambda \notin \{1, 3\}$ are in one-to-one correspondence with non-trivial holomorphic vector-valued functions $W_{\mathcal{M}}$ on $\mathcal{M}(R_3)$ possessing the symmetries* (34).

§ 5. Projective structures on $\mathcal{M}(R_3)$

A projective structure [1], [10]–[13] on a Riemann surface $\mathcal{M}$ is a multivalued meromorphic function $p(\widetilde{m})$ that transforms linear fractionally on going around closed paths in $\mathcal{M}$. The corresponding homeomorphism $\pi_1(\mathcal{M}) \to PSL_2(\mathbb{C})$ is called the monodromy of the structure. A classical example, which goes back to Poincaré, is a Fuchsian projective structure mapping the universal cover of a hyperbolic surface $\mathcal{M}$ onto the unit disc. A projective structure $p(\widetilde{m})$ with critical points is called a branched structure. For such a structure one can define the branching divisor

$$\mathsf{D}(p) := \sum_{m\in\mathcal{M}} (k(p,m)-1)\cdot m. \tag{35}$$

Here $k(p,m)$ is the branching index of p at a point m, and the degree $\deg \mathsf{D}(p)$ is called the (total) branching number of the projective structure.

We show in this section that each eigenvalue-eigenvector pair (λ, u) of the PS_3 integral equation is related to the existence on the surface $\mathcal{M}(R_3)$ of a projective structure of a special form.

5.1. Invariant of monodromy group. The key factor relating the PS_3 equations to projective structures is that the monodromy $\mathbf{T}$ defined by equalities (14) has a quadratic invariant

$$J(W) := \sum_{k=1}^{n} W_k^2 - \delta \sum_{j<s}^{n} W_j W_s. \tag{36}$$

For $n=3$ the form J is non-degenerate if $-2 \ne \delta \ne 1$. The value of J on the solution $W(\widetilde{y})$ of the Riemann problem is constant because this solution is bounded in the neighbourhood of the punctures of $\mathcal{Y}$. Hence for $0 \ne \lambda \ne 3$ the solution W either ranges in the non-degenerate quadric $\{J(W)=J_0\}$ or in the cone $\{J(W)=0\}$.

5.2. Geometry of quadric and cone.

5.2.1. Coordinates $p^{\pm}$ on a quadric. On a non-degenerate projective quadric $\{J(W)=J_0\}$ we introduce global coordinates p^+ and p^- ranging in the Riemann sphere. Recall that a non-degenerate quadric contains two families of linear elements, which we shall conveniently denote by the signs '+' and '−'. Distinct lines in one family are disjoint and two lines from distinct families must intersect. For an arbitrary point $W := (W_1, W_2, W_3)^t$ in the quadric we consider the pair of lines through this point; they intersect with the plane at infinity at the points

$$W^{\pm}(W) = (W_1^{\pm} : W_2^{\pm} : W_3^{\pm})^t \sim \begin{pmatrix} \tau\Sigma W_1 \pm (W_2 - W_3) + \tau^{-1} \\ \tau\Sigma W_2 \pm (W_3 - W_1) + \tau^{-1} \\ \tau\Sigma W_3 \pm (W_1 - W_2) + \tau^{-1} \end{pmatrix} \in \mathbb{CP}_2,$$
$$\Sigma := W_1 + W_2 + W_3, \qquad \tau := \sqrt{\frac{\delta-1}{J_0}}, \tag{37}$$

which lie in the non-degenerate conic at infinity

$$\mathcal{C} := \{(W_1 : W_2 : W_3)^t \in \mathbb{CP}_2 : J(W) = 0\}. \tag{38}$$

Conversely, corresponding to each pair of points $W^\pm$ in the conic $\mathcal{C}$ there is a unique point W in the projective quadric. It lies at the intersection of the element of the '+' family passing through W^+ and the element of the '−' family passing through W^-. The homogeneous coordinates of W are bilinear forms of the homogeneous coordinates of W^+ and W^-, but their explicit expressions are too lengthy and we do not present them here.

Identifying the conic $\mathcal{C}$ and a projective line by means of a stereographic projection we can associate each point in the projective quadric with an ordered pair of complex 'numbers' $p^\pm \in \mathbb{CP}_1$, the stereographic coordinates of the points $W^\pm$. The part of the quadric lying at infinity ($= \mathcal{C}$) corresponds to the diagonal $\{p^+ = p^-\}$ of $\mathbb{CP}_1 \times \mathbb{CP}_1$.

The linear transformations $\mathbf{T}$ preserving J form the complex group $O_3(J)$ and act in the natural way in the conic at infinity $\mathcal{C} \cong \mathbb{CP}_1$. This action defines, up to conjugation, a (spinor) representation in the group of linear fractional maps:

$$\chi\colon O_3(J) \to PSL_2(\mathbb{C}). \tag{39}$$

The transformation of the coordinates $p^\pm$ on the quadric under the action of $\mathbf{T} \in O_3(J)$ depends on whether $\mathbf{T}$ preserves the '±' families of linear elements or transforms them into each other:

$$\begin{aligned} p^\pm(\mathbf{T}W) &= \chi(\mathbf{T})p^\pm(W), \qquad && \mathbf{T} \in SO_3(J), \\ p^\pm(\mathbf{T}W) &= \chi(\mathbf{T})p^\mp(W), \qquad && \mathbf{T} \in O_3(J) \setminus SO_3(J). \end{aligned} \tag{40}$$

For explicit formulae of the stereographic coordinate $p(W)$ and the linear fractional map $\chi(\mathbf{T})$ we bring $J(W)$ to the form $J_\bullet(V) := V_1V_3 - V_2^2$ (which is convenient for calculations) by the map $W = \mathbf{K}V$ with matrix

$$\begin{gathered} \mathbf{K} := (3\delta+6)^{-1/2} \begin{Vmatrix} 1 & 1 & 1 \\ 1 & \varepsilon^2 & \varepsilon \\ 1 & \varepsilon & \varepsilon^2 \end{Vmatrix} \cdot \begin{Vmatrix} 0 & \mu^{-1} & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{Vmatrix}, \\ \varepsilon := \exp(2\pi i/3), \qquad \mu := \sqrt{\frac{\delta-1}{\delta+2}} = \sqrt{\frac{3-\lambda}{2\lambda}}\,. \end{gathered} \tag{41}$$

The isomorphism $(W_1 : W_2 : W_3)^t \in \mathcal{C} \cong \mathbb{CP}_1 \ni p$ gives us the formulae

$$p(W) := \frac{V_2(W)}{V_1(W)} = \frac{V_3(W)}{V_2(W)} = \mu\frac{W_1+W_2+W_3}{W_1+\varepsilon^2W_2+\varepsilon W_3} = \mu^{-1}\frac{W_1+\varepsilon W_2+\varepsilon^2W_3}{W_1+W_2+W_3}, \tag{42}$$

$$(W_1 : W_2 : W_3)^t(p) \sim \begin{pmatrix} p^2\mu + p + \mu \\ p^2\mu\varepsilon^2 + p + \mu\varepsilon \\ p^2\mu\varepsilon + p + \mu\varepsilon^2 \end{pmatrix}. \tag{43}$$

Fixing stereographic coordinates on the conic defines completely the homomorphism (39); its further properties are collected in the following lemma.

Lemma 12. (i) *The action of the (pseudo-)orthogonal group $O_3(J)$ defines a representation χ in* (39) *such that*

$$p(\mathbf{T}W) = \chi(\mathbf{T})p(W), \qquad W = (W_1 : W_2 : W_3)^t \in \mathcal{C}, \quad \mathbf{T} \in O_3(J). \tag{44}$$

The restriction of χ to the connected component $SO_3(J)$ is an isomorphism.

(ii) *The action of $O_3(J)$ in the cone $\{J(W) = 0\}$ in $\mathbb{C}^3$ satisfies the identity*

$$(\det \mathbf{T})\mathbf{TK}\begin{pmatrix} 1 \\ p \\ p^2 \end{pmatrix} = \left[\frac{d}{dp}\chi(\mathbf{T})p\right]^{-1}\mathbf{K}\begin{pmatrix} 1 \\ \chi(\mathbf{T})p \\ (\chi(\mathbf{T})p)^2 \end{pmatrix}, \qquad \mathbf{T} \in O_3(J), \quad p \in \mathbb{C}, \quad \chi(\,\cdot\,) \in PSL_2(\mathbb{C}), \tag{45}$$

which makes it possible to recover the expression for $\chi(\mathbf{T})$. In particular,

$$\begin{aligned} \chi(\mathbf{D}_0)p &= \varepsilon p, \quad \varepsilon := \exp\left(\frac{2\pi i}{3}\right), \\ \chi(\mathbf{D}_1)p &= \frac{1}{p}, \\ \chi(\mathbf{D})p &= \frac{\mu p - 1}{p - \mu}, \quad \mu := \sqrt{\frac{\delta - 1}{\delta + 2}} = \sqrt{\frac{3 - \lambda}{2\lambda}}. \end{aligned} \tag{46}$$

Proof. The matrix $\mathbf{K}^{-1}\mathbf{TK}$ is $J_\bullet$-orthogonal and takes a vector $V := (1, p, p^2)^t$ in the cone $\{J_\bullet(V) = 0\}$ to a vector $(P_1(p), P_2(p), P_3(p))^t$ in the same cone, where the $P_k(p)$ are polynomials of degree at most 2. Each zero of the polynomial $P_2(p)$ is simple and it is also a zero — moreover, a double zero — of precisely one of the polynomials $P_1(p)$ and $P_3(p)$. This follows from the non-singularity of the matrix $\mathbf{K}^{-1}\mathbf{TK}$. The image of V can be uniquely, up to the simultaneous change of the signs of the complex numbers a, b, c, and d, represented in the following form:

$$\mathbf{K}^{-1}\mathbf{TK}V = \begin{pmatrix} (cp + d)^2 \\ (ap + b)(cp + d) \\ (ap + b)^2 \end{pmatrix} = (ad - bc)\left[\frac{d}{dp}\chi(\mathbf{T}, p)\right]^{-1}\begin{pmatrix} 1 \\ \chi(\mathbf{T}, p) \\ \chi^2(\mathbf{T}, p) \end{pmatrix}, \tag{47}$$

where we set by definition $\chi(\mathbf{T}, p) := \dfrac{ap + b}{cp + d} =: \chi(\mathbf{T})p$.

We claim that $\det \mathbf{T} = ad - bc$. A direct calculation making use of (47) shows that $\det \mathbf{T} = (ad - bc)^3$. The equality $(ad - bc)^2 = 1$ is a consequence of the (pseudo-)orthogonal invariance of the bilinear form $J_\bullet(\,\cdot\,, \cdot\,)$ polar to the quadratic form $J_\bullet(\,\cdot\,)$. Let $V^{(1)} := (1, p_1, p_1^2)^t$ and $V^{(2)} := (1, p_2, p_2^2)^t$ be vectors in the cone $\{J_\bullet(V) = 0\}$. Then

$$\begin{aligned} J_\bullet(\mathbf{K}^{-1}\mathbf{TK}V^{(1)}, \mathbf{K}^{-1}\mathbf{TK}V^{(2)}) &= (ad - bc)^2\frac{(\chi(\mathbf{T}, p_1) - \chi(\mathbf{T}, p_2))^2}{\dot\chi(\mathbf{T}, p_1)\dot\chi(\mathbf{T}, p_2)} \\ &= (ad - bc)^2(p_1 - p_2)^2 \\ &= (ad - bc)^2 J_\bullet(V^{(1)}, V^{(2)}) \\ &= J_\bullet(V^{(1)}, V^{(2)}). \end{aligned} \tag{48}$$

We have thus established (45) for the above-defined linear fractional map χ. We now prove the transformation law (44) for the stereographic coordinate (42). The point $W \sim \mathbf{K}(1 : p : p^2)^t \in \mathcal{C}$ has the coordinate p and $\mathbf{T}W \sim \mathbf{K}(1 : \chi(\mathbf{T}, p) : \chi^2(\mathbf{T}, p))^t$ has the coordinate $\chi(\mathbf{T}, p)$, so that (44) holds.

It remains to show that χ is an isomorphism between $SO_3(J)$ and PSL_2. The injectivity of χ follows by (45). We now discuss its surjectivity. Each $\chi \in SL_2$ defines by formula (47) a matrix $\mathbf{K}^{-1}\mathbf{T}\mathbf{K}$, which we claim to be $J_\bullet$-orthogonal. It suffices to show that this matrix preserves the quadratic form $J_\bullet(\,\cdot\,)$ and the bilinear form $J_\bullet(\,\cdot\,,\cdot\,)$ for the vectors in the cone $\{J_\bullet(V) = 0\}$. The first is obvious by (47), and the second by (48). The equality $\det \mathbf{T} = (ad - bc)^3 = 1$ has already been established.

5.2.2. Coordinate p and form ω_J on a cone. Unfortunately, one cannot introduce two global coordinate functions on the cone $\{W \in \mathbb{C}^3 : J(W) = 0\}$. Nevertheless, projecting the cone from the origin onto the conic $\mathcal{C}$ at infinity we can for $W \neq 0$ define the coordinate function $p(W)$ by formula (42). The role of the second 'coordinate function' in our analysis of the monodromy problem will be taken by the 1-form furnished by the following result.

Lemma 13. (i) *Let $W \neq 0$ be a point in the cone $\{J(W) = 0\}$ and let Z be a vector tangent to the cone at W. Then the form ω_J defined as the ratio of two collinear vectors*

$$\langle \omega_J(W), Z\rangle := (W \times Z) : \nabla J(W) \tag{49}$$

($\times$ *is the vector product in* $\mathbb{C}^3$ *and* ∇ *is the gradient operator*) *is holomorphic for* $W \neq 0$.

(ii) *The form ω_J is invariant under motions of the cone*:

$$\omega_J(\mathbf{T}W) = (\det \mathbf{T})\omega_J(W), \qquad \mathbf{T} \in O_3(J). \tag{50}$$

(iii) *The forms ω_J and dp are proportional*:

$$(\det \mathbf{K}) W\, dp(W) = \mathbf{K} \begin{pmatrix} 1 \\ p(W) \\ p^2(W) \end{pmatrix} \omega_J; \tag{51}$$

here $W \neq 0$ is a point in the cone and $\mathbf{K}$ is the matrix (41) *taking J to the form $J_\bullet$.*

Proof. (i) We prove that the vectors in the definition of ω_J are collinear:

$$\nabla J(W) \times (W \times Z) = 2W J(W, Z) - 2Z J(W) = 0;$$

here $J(\,\cdot\,,\cdot\,)$ is the polar bilinear form of the quadratic form $J(\,\cdot\,)$. The form ω_J has the coordinate representation

$$\omega_J(W) = \frac{W_1 dW_2 - W_2 dW_1}{2W_3 - \delta W_1 - \delta W_2} = \frac{W_2 dW_3 - W_3 dW_2}{2W_1 - \delta W_2 - \delta W_3} = \frac{W_3 dW_1 - W_1 dW_3}{2W_2 - \delta W_3 - \delta W_1}\,, \tag{52}$$

which shows that this form is holomorphic for $W \neq 0$.

(ii) We have

$$\begin{aligned}\langle\omega_J(\mathbf{T}W),\mathbf{T}Z\rangle &:= (\mathbf{T}W\times\mathbf{T}Z):\nabla J(\mathbf{T}W)\\ &= (\det\mathbf{T})\mathbf{T}^{-1t}(W\times Z):\mathbf{T}^{-1t}\nabla J(W)\\ &= (\det\mathbf{T})(W\times Z):\nabla J(W) =: (\det\mathbf{T})\langle\omega_J(W),Z\rangle.\end{aligned}$$

(iii) Formula (51) holds for each non-degenerate form $J(W)$ if the stereographic coordinate p on the cone is selected so as to conform with the matrix $\mathbf{K}$ taking J to the form $J_\bullet(V)$, that is, $p(W)=V_2(W)/V_1(W)$ and $W=\mathbf{K}V$. For $J=J_\bullet$ the equality can be immediately verified:

$$(V_1,V_2,V_3)^t d(V_3/V_2) = \big((V_2/V_3)dV_3 - dV_2\big)\big(1, V_3/V_2, (V_3/V_2)^2\big)^t.$$

In the case of an arbitrary form J we make the linear change $\mathbf{K}V=W$ and keep the old definition of the stereographic coordinate p. Taking account of the identity $\omega_J(\mathbf{K}V)=(\det\mathbf{K})\omega_{J_\bullet}(V)$ we arrive at relation (51).

5.3. Solution on a non-degenerate quadric. In this subsection we prove the following result.

Theorem 4. *For $\lambda\notin\{0,1,3\}$ the eigenvalue-eigenvector pairs $(\lambda,u(t))$ of the PS_3 integral equation with non-trivial invariant J_0 are in one-to-one correspondence with projective structures $p(\widetilde{m})$ on $\mathcal{M}(R_3)$ with total branching number $2g(\mathcal{M})-2$ such that the symmetries*

$$p([\mathbf{r}]\cdot\widetilde{m}) = \chi\circ\mathbf{T}([\mathbf{r}])p(\widetilde{m}) \tag{53}$$

hold for all $\widetilde{m}\in\widetilde{\mathcal{M}}$ and all even classes $[\mathbf{r}]\in\pi_1(\mathcal{Y})$ and fail at least at one point $\widetilde{m}$ for some odd class $[\mathbf{r}]$.

Remark. By Lemma 10 the existence of the symmetries (53) for all even loops $[\mathbf{r}]$ indicates that the monodromy of the structure is $\chi\circ\mathbf{T}_{\mathcal{M}}$ and there exist one (cases $\mathbf{2}_5\mathbf{3}_0$, $\mathbf{2}_4\mathbf{3}_1$, $\mathbf{2}_4\mathbf{3}_0$, $\mathbf{2}_2\mathbf{3}_2$) or two (case $\mathbf{2}_3\mathbf{3}_1$) additional symmetries. In case $\mathbf{2}_6\mathbf{3}_0$ of an equation in general position there are no additional symmetries.

Proof of Theorem 4. (1) *The correspondence* $(\lambda,u)\to p$. Let $(\lambda,u(t))$ be an 'eigenpair' of a PS_3 of the required form. By Theorem 3 it corresponds to an analytic vector-valued function $W_{\mathcal{M}}(\widetilde{m})$ on $\mathcal{M}(R_3)$ ranging in the non-degenerate quadric $\{J(W)=J_0\}$. This vector defines by formulae (37) and (42) a pair of meromorphic functions $p^\pm(\widetilde{m})$ on $\widetilde{\mathcal{M}}$ whose symmetries (34) can be expressed as follows:

$$p^\pm([\mathbf{r}]\cdot\widetilde{m}) = \chi\circ\mathbf{T}([\mathbf{r}])p^\pm(\widetilde{m}), \qquad [\mathbf{r}]\in\ker\mathrm{sign}\subset\pi_1(\mathcal{Y}), \tag{54}$$

$$p^\pm([\mathbf{r}]\cdot\widetilde{m}) = \chi\circ\mathbf{T}([\mathbf{r}])p^\mp(\widetilde{m}), \qquad [\mathbf{r}]\in\pi_1(\mathcal{Y})\setminus\ker\mathrm{sign}. \tag{55}$$

The last equality shows how one can recover one of the structures $p^\pm$ from the other. In particular, the branching divisor $\mathsf{D}(p^+)$ can be obtained from $\mathsf{D}(p^-)$ by an arbitrary covering transformation of $\mathcal{M}$.

We claim that both structures $p^{\pm}$ have branching number $2g(\mathcal{M})-2$. For a proof we consider the (Klein) quadratic differential

$$\Omega(m)=\frac{dp^{+}(\widetilde{m})dp^{-}(\widetilde{m})}{(p^{+}(\widetilde{m})-p^{-}(\widetilde{m}))^{2}}, \qquad m\in\mathcal{M}, \quad \widetilde{m}\in\widetilde{\mathcal{M}}. \tag{56}$$

This expression is the infinitesimal form of the cross-ratio of four points and is therefore preserved by simultaneous linear fractional transformations of the variables p^{+} and p^{-} (cf. equality (48)). The holomorphy of $W_{\mathcal{M}}$ is equivalent to the inequality $p^{+}(\widetilde{m})\neq p^{-}(\widetilde{m})$ everywhere on $\widetilde{\mathcal{M}}$, therefore for each point $\widetilde{m}$ one can find a linear fractional change of variables taking $p^{\pm}$ to the values $p^{+}(\widetilde{m})=1$ and $p^{-}(\widetilde{m})=0$. Expanding now the functions in the neighbourhood of $\widetilde{m}$ in powers of the local variable we see that

$$\mathsf{D}(p^{+})+\mathsf{D}(p^{-})=(\Omega).$$

On the other hand, the degree of the zeros of a quadratic differential is $4g(\mathcal{M})-4$.

We take the projective structure $p^{+}(\widetilde{m})$ for the required structure p. It has branching number $2g-2$, and equalities (53) hold for all even $[\mathbf{r}]$, while for odd $[\mathbf{r}]$ the symmetry (53) can be expressed as the equality $p^{+}(\widetilde{m})=p^{-}(\widetilde{m})$, and it *fails* for each $\widetilde{m}$. Note that the structure $p^{-}(\widetilde{m})$ has the same properties.

(2) *The correspondence* $p\to(\lambda,u)$. Let $p(\widetilde{m})$ be a projective structure having all the properties listed in the theorem. We set $p^{+}(\widetilde{m}):=p(\widetilde{m})$. Then equality (55) defines a new projective structure p^{-}. It is easy to verify that the definition of this new structure is independent of one's choice of the even class $[\mathbf{r}]$, and the transformation law for p^{-} is as in (54).

We claim that $p^{+}(\widetilde{m})\neq p^{-}(\widetilde{m})$ everywhere. By the assumptions of the theorem $p^{+}(\widetilde{m})\not\equiv p^{-}(\widetilde{m})$, therefore (56) defines a meromorphic quadratic differential on $\mathcal{M}$. Reducing the functions $p^{\pm}$ locally to a form convenient for estimates we can see that

$$\mathsf{D}(p^{+})+\mathsf{D}(p^{-})\geqslant(\Omega),$$

where the strict inequality indicates the existence of points at which $p^{+}=p^{-}$. However,

$$\deg\mathsf{D}(p^{+})+\deg\mathsf{D}(p^{-})=4g(\mathcal{M})-4=\deg(\Omega),$$

so that there can be no such points.

Recovering the points $W^{\pm}$ in the conic $\mathcal{C}$ from their stereographic coordinates (43) and recovering the point in the quadric from these points in the conic (we use here the invariant J_0) we obtain from $p^{\pm}(\widetilde{m})$ a holomorphic vector-valued function $W_{\mathcal{M}}$. The transformation laws (54), (55) for the pair of structures become the law (34) for the transformations of $W_{\mathcal{M}}$. By Theorem 3 this vector corresponds to an 'eigenpair' of the PS_3 integral equation.

Remark 1. It is clear from the proof of the theorem that there exist in fact two correspondences $(\lambda,u)\leftrightarrow p$ associating an eigenpair of PS_3 with a special projective structure: the one taking the pair (λ,u) to the structure p^{+}, and the other taking it to p^{-}. The transposition of subscripts of $p^{\pm}$ results in a change of sign of the eigenfunction $u(t)$.

Remark 2. It follows from the symmetries (53) of the projective structure p that the branching divisor of p is invariant with respect to even covering transformations $[\mathbf{r}] \in \ker \operatorname{sign}/pr_*\pi_1(\mathcal{M}_\circ)$ of $\mathcal{M}$. For instance, in cases $\mathbf{2}_3\mathbf{3}_1$ and $\mathbf{2}_2\mathbf{3}_2$ the divisor $\mathsf{D}(p)$ contains two (of the four) fixed points of a rotation of $\mathcal{M}$ of order 3 (see Fig. 11).

Remark 3. In the case $\mathbf{2}_4\mathbf{3}_0$ the surface $\mathcal{M}$ is a torus and the projective structure in it is — up to a linear fractional transformation — the exponential function of an Abelian integral of the first kind. In this case the solutions can be expressed by explicit formulae, which was accomplished in [8] by other means.

5.4. Solution on a cone.

Theorem 5. *For $\lambda \notin \{0,1,3\}$ the eigenvalue-eigenvector pairs $(\lambda, u(t))$ of the* PS_3 *integral equation with non-trivial invariant J_0 are in one-to-one correspondence with pairs consisting of a projective structure $p(\widetilde{m})$ and a non-trivial holomorphic differential $\omega(m)$ on $\mathcal{M}(R_3)$ possessing the symmetries*

$$p([\mathbf{r}]\cdot\widetilde{m}) = \chi\circ\mathbf{T}([\mathbf{r}])p(\widetilde{m}), \quad \widetilde{m}\in\widetilde{\mathcal{M}}, \quad [\mathbf{r}]\in\pi_1(\mathcal{Y}), \tag{57}$$

$$\omega([\mathbf{r}]\cdot m) = \operatorname{sign}([\mathbf{r}])\omega(m), \qquad m\in\mathcal{M}, \quad [\mathbf{r}]\in\pi_1(\mathcal{Y})/pr_*\pi_1(\mathcal{M}_\circ), \tag{58}$$

and with divisors related by the inequality

$$\mathsf{D}(p) \leqslant (\omega). \tag{59}$$

Proof. (1) *The correspondence* $(\lambda,u)\to(p,\omega)$. Let $(\lambda,u(t))$ be an 'eigenpair' of the PS_3 equation such that the holomorphic vector-valued function $W_{\mathcal{M}}$ corresponding to it by Lemma 3 has the invariant J_0 equal to zero. The solution $W_{\mathcal{M}}$ transplants the function p and the holomorphic differential ω_J from the cone $\{J(W)=0\}$ to $\mathcal{M}$. Since $W_{\mathcal{M}}$ is not identically equal to zero, formulae (42) and (52) define the meromorphic function $p(\widetilde{m})$ and the holomorphic differential $\omega(m)$ also at the zeros of the vector field. The form ω on the surface inherits the transformation law (50) of the form ω_J on the cone; the transformation law for $p(\widetilde{m})$ can be found from the symmetries (34) of $W_{\mathcal{M}}$.

Finally we claim that the branching divisor of $p(\widetilde{m})$ is not greater than the zero divisor of ω. It is clear from (51) that the functions ω/dp and $\omega/d(1/p)$ are holomorphic because they are linear combinations of the components W_1, W_2, and W_3 of the solution. Hence the local branching number of p is not greater than the order of the zero of the differential ω.

(2) *The correspondence* $(p,\omega)\to(\lambda,u)$. Conversely, let $p(\widetilde{m})$ be a projective structure with symmetries (57) and let $\omega(m)$ be a non-trivial symmetric holomorphic differential on $\mathcal{M}$. We set by definition

$$W_{\mathcal{M}}(\widetilde{m}) := (\det\mathbf{K})^{-1}\frac{\omega(\widetilde{m})}{dp(\widetilde{m})}\mathbf{K}\begin{pmatrix}1\\ p(\widetilde{m})\\ p^2(\widetilde{m})\end{pmatrix}. \tag{60}$$

This is a holomorphic vector-valued function of $\widetilde{m}$ in view of relation (59) between the branching divisor of p and the zero divisor of ω. The vector-valued function $W_{\mathcal{M}}$ satisfies the required transformation law:

$$\begin{aligned} W_{\mathcal{M}}([\mathbf{r}]\cdot\widetilde{m}) &= \bigl(\dot\chi(T([\mathbf{r}]),p(\widetilde{m}))\bigr)^{-1}\operatorname{sign}([\mathbf{r}])(\det\mathbf{K})^{-1} \\ &\quad\times\frac{\omega(\widetilde{m})}{dp(\widetilde{m})}\mathbf{K}\begin{pmatrix}1\\ \chi\circ\mathbf{T}([\mathbf{r}])p(\widetilde{m})\\ (\chi\circ\mathbf{T}([\mathbf{r}])p(\widetilde{m}))^2\end{pmatrix} \\ &\overset{(45)}{=} (\det\mathbf{K})^{-1}\frac{\omega(\widetilde{m})}{dp(\widetilde{m})}\mathbf{T}([\mathbf{r}])\mathbf{K}\begin{pmatrix}1\\ p(\widetilde{m})\\ p^2(\widetilde{m})\end{pmatrix} := \mathbf{T}([\mathbf{r}])W_{\mathcal{M}}(\widetilde{m}). \end{aligned}$$

By Theorem 3 the pair (p,ω) generates an 'eigenpair' (λ,u) of the PS_3 equation.

Applying to the vector $W_{\mathcal{M}}(\widetilde{m})$ so constructed the procedure from part (1) of the proof we return to the original projective structure p and the holomorphic differential ω; this is a consequence of (51). Hence the correspondences in parts (1) and (2) are reciprocal.

Remark. Symmetric holomorphic forms on the surface are described by Lemma 11. It follows from it, in particular, that PS_3 equations of types $\mathbf{2}_4\mathbf{3}_0$, $\mathbf{2}_3\mathbf{3}_1$, and $\mathbf{2}_2\mathbf{3}_2$ do not have solutions with invariant J_0 equal to zero.

§ 6. Applications

In this section we present applications of the complex geometric theory of the PS_3 integral equation developed above. For instance, combining the results of Theorems 4 and 5 we obtain a test for eigenvalues of generic equations.

Theorem 6. *A complex number $\lambda\notin\{0,1,3\}$ is an eigenvalue of a PS_3 equation of type $\mathbf{2}_6\mathbf{3}_0$ if and only if there exists on the Riemann surface $\mathcal{M}(R_3)$ a projective structure without branching or of total branching number* 2 *that has monodromy $\chi\circ\mathbf{T}_{\mathcal{M}}$, which depends on λ as a parameter. Structures without branching correspond to eigenvalues of multiplicity* 2.

Proof. (1) If (λ,u) is an 'eigenpair' of a typical PS_3 equation, then there exist by Theorems 4 and 5 a projective structure $p(\widetilde{m})$ on $\mathcal{M}(R_3)$ with the required monodromy and of branching number at most 2. Let $\deg\mathsf{D}(p)=1$. Then, in view of the symmetry (57) in the case of an odd loop $[\mathbf{r}]$, the point $\mathsf{D}(p)$ is a fixed point of the hyperelliptic involution of $\mathcal{M}$. We choose in a neighbourhood of $\mathsf{D}(p)$ a local coordinate function w changing sign under the involution. Taking an appropriate linear fractional transformation of p, in the neighbourhood of a branch point the symmetry (57) for odd loops can be expressed as $p(-w)=-p(w)$, therefore $\deg\mathsf{D}(p)$ is an even integer. Another reason for this evenness of the branching number is the fact that the monodromy $\mathbf{T}_{\mathcal{M}}$ is trivial for half the cycles in the canonical basis of $\pi_1(\mathcal{M})$, and therefore $\chi\circ\mathbf{T}_{\mathcal{M}}$ can be lifted to a representation $\pi_1(\mathcal{M})\to SL_2$. See [11] for greater detail.

(2) Conversely, let p be a projective structure on $\mathcal{M}$ with monodromy $\chi\circ\mathbf{T}_{\mathcal{M}}$. By Lemma 10, for a typical equation this means that equality (53) holds for all even

paths $[\mathbf{r}]$. Let the branching number of the structure be 2. If the symmetry (53) fails for odd loops, then λ is an eigenvalue by Theorem 4. On the other hand if such symmetry holds, then the branching divisor of p is invariant under the hyperelliptic involution and therefore it is equal to the zero divisor of some holomorphic form ω (which, of course, changes sign under the involution). In that case λ is an eigenvalue by Theorem 5.

Now let p be a non-branched structure and let $[\mathbf{r}]$ be an arbitrary odd loop. Then the two projective structures $p(\widetilde{m})$ and $p_1(\widetilde{m}) := \chi\circ\mathbf{T}([\mathbf{r}]^{-1})p([\mathbf{r}]\cdot\widetilde{m})$ have the same monodromy and are not branched. By Poincaré's theorem these structures are the same, that is, the symmetry (57) holds also for odd loops $[\mathbf{r}]$. We can take for ω an arbitrary holomorphic differential on $\mathcal{M}$, therefore a non-branched projective structure generates a 2-dimensional eigenspace of the PS_3 equation. There exist no other eigenvectors in this case because by Poincaré's theorem [12], [13] two projective structures with the same monodromy and of total branching number smaller than $4g(\mathcal{M})-4$ must be the same.

We now explain how the spectrum of a typical PS_3 equation can be geometrically described. The space $\mathcal{P}_2(\mathcal{M})$ of projective structures on $\mathcal{M}(R_3)$ with a pair of branch points (see [11]–[13] for the definitions) is a 5-dimensional complex variety. The map assigning to a projective structure its monodromy embeds $\mathcal{P}_2(\mathcal{M})$ in the 6-dimensional complex space of monodromies $\mathrm{Hom}(\pi_1(\mathcal{M}), PSL_2)/PSL_2$. In this space the representation $\chi\circ\mathbf{T}_{\mathcal{M}}$ defines a complex curve parametrized by the spectral parameter λ. The intersection points of the curve and the 5-dimensional image of $\mathcal{P}_2(\mathcal{M})$ correspond to the eigenvalues of the PS_3 equation. The space $\mathcal{P}_2^s(\mathcal{M})$ of structures with additional symmetry corresponding to the hyperelliptic involution $\mathcal{M}$ has complex dimension 4. One would conjecture that the image of $\mathcal{P}_2^s(\mathcal{M})$ in the space of monodromies intersects the curve defined by the representation $\chi\circ\mathbf{T}_{\mathcal{M}}$ only for surfaces $\mathcal{M}$ lying in some variety of codimension 1 in the Teichmüller space $\mathcal{T}(2)$. The points $\mathcal{M}$ in the Teichmüller space that carry the required non-branched structures are even rarer.

6.1. Localization of the spectrum. The operator approach to the analysis of PS equations [2], [4] gives one upper and lower bounds for the eigenvalues λ of an equation. These bounds depend on metric properties of the transformation $R(t)$. The following result establishes uniform bounds for the spectrum of a PS equation for all rational functions of degree 3.

Theorem 7. *The spectrum of an arbitrary PS integral equation with rational transformation R_3 of degree* 3 *lies in the interval* $[0,3]$.

Proof. We use one fact from the operator analysis of the PS equation [2], [4]: the spectrum lies on the real axis. (It is useful to interpret this fact in geometric language.) For real $\lambda\notin[0,3]$, as seen from Lemma 12 (see (46)), the monodromy $\chi\circ\mathbf{T}_{\mathcal{M}}$ is unitary. In view of Theorems 4 and 5 we can complete the proof using the following result.

Lemma 14. *If the branching number of a projective structure p on a Riemann surface $\mathcal{M}$ of genus g is at most $2g-2$, then the monodromy p is not unitary.*

Proof. Poincaré has proved this result for non-branched structures. The same idea can be applied in the case when the branching number of p is not greater than the absolute value of the Euler characteristic of $\mathcal{M}$. Assume that the monodromy of p is unitary. Then the expression

$$ds^2 = (1+p\overline{p})^{-2}\,dp\,d\overline{p}$$

defines a conformal metric on $\mathcal{M}$ that is degenerate at the points in the branching divisor. The Gaussian curvature form of this metric,

$$\Theta = -(2\pi i)^{-1} d'' d' \ln(1+p\overline{p})^{-2} |dp/dx|^2,$$

where x is the local variable, is positive in $\mathcal{M} \setminus |\mathsf{D}(p)|$. On the other hand, by the Gauss–Bonnet formula we obtain

$$0 < \int_{\mathcal{M}} \Theta \overset{(*)}{=} 2 - 2g + \deg \mathsf{D}(p) \leqslant 0.$$

For reasons of space we have left out the intermediate calculations $(*)$ (see [14]).

6.2. Representation of solutions of a PS_3 equation. One geometric way to define projecting structures, used already by Klein, is as follows. We cut the surface $\mathcal{M}$ with linearly polymorphic function p on it so that in the resulting (not necessarily simply connected) domain we could find a single-valued branch of p; the p-image of the cut surface is a surface $\mathcal{F}$ which lies (possibly in a non-schlicht manner, forming several sheets) over the Riemann sphere. The boundary components of $\mathcal{F}$ are organized into pairs $(\partial\mathcal{F})_s^+$, $(\partial\mathcal{F})_s^-$, $s = 1, 2, \dots$, each associated with some linear fractional map $(\partial\mathcal{F})_s^+ \to (\partial\mathcal{F})_s^-$ from the monodromy group of p that reverses the natural orientation of the boundary. Such surfaces $\mathcal{F}$ are called *membranes*. As an example we construct below a membrane for a projective structure giving rise to an 'eigenpair' of a PS_3 equation.

6.2.1. Membrane. Let λ be a real number in the interval $(1, 2)$. The fixed points of the second-order rotation $\chi(\mathbf{D})$ depending on the parameter λ are complex conjugate and lie on the arc $\{p \in \mathbb{C} : |p| = 1,\ |\arg p| < \pi/3\}$ of the unit circle. We consider the circle C passing through the fixed points of $\chi(\mathbf{D})$ and orthogonal to the unit circle. In the complex p-plane the circle C and the rotated real axis $\varepsilon\widehat{\mathbb{R}}$ bound an annulus-type domain $\mathcal{F}_1$; another domain of this kind $\mathcal{F}_2$ is bounded by the real axis $\widehat{\mathbb{R}}$ and the circle εC. The two-sheeted membrane $\mathcal{F}$ consists of the annuli $\mathcal{F}_1$ and $\mathcal{F}_2$ glued crosswise along a cut Γ symmetric relative to the unit circle (see Fig. 12).

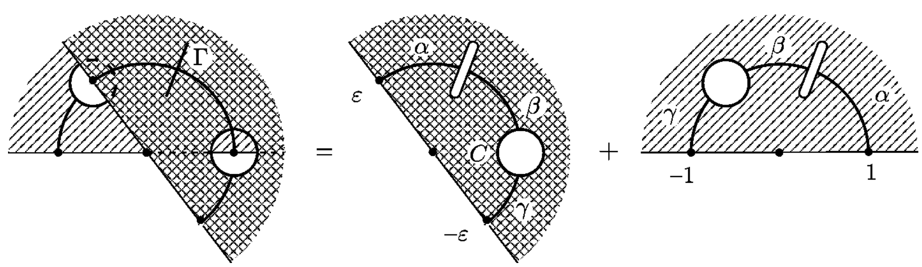


Figure 12. Membrane $\mathcal{F}$ in the complex p-plane

6.2.2. Real algebraic curve. Identifying parts of the boundary of the '4-connected' domain $\mathcal{F}$ by means of the orientation-reversing linear fractional maps

$$\begin{aligned}\chi(\mathbf{D}_2\mathbf{D}_3)&: \widehat{\mathbb{R}} \to \varepsilon\widehat{\mathbb{R}},\\ \chi(\mathbf{D}_3\mathbf{D})&: C \to \varepsilon C,\end{aligned} \tag{61}$$

we make from $\mathcal{F}$ a compact Riemann surface $\mathcal{M}_* = \mathcal{M}_*(\lambda, \partial\Gamma)$. It depends on three real parameters and is hyperelliptic because it has genus $g = 2$. One can explicitly describe an anticonformal involution $\overline{H}$ of this surface such that the algebraic curve $\mathcal{M}_*$ is a real curve with respect to it. Namely, the reflection $\overline{H}$ acts on each sheet $\mathcal{F}_1$ and $\mathcal{F}_2$ as the inversion $p \to 1/\overline{p}$ relative to the unit circle. The action of $\overline{H}$ is well defined on Γ and preserves the identifications (61) of the boundary components.

There exists on $\mathcal{M}_*$ a (unique, up to a real linear fractional transformation) second-order element $y(m) \in \mathbb{C}(\mathcal{M}_*)$ such that its reflection is equal to its complex conjugation:

$$y(\overline{H}m) = \overline{y}(m), \qquad m \in \mathcal{M}_*. \tag{62}$$

If the cut Γ is transversal to the unit circle, then the fixed points of $\overline{H}$ form three ovals α, β, and γ, as in Fig. 12. The images $y(\alpha)$, $y(\beta)$, and $y(\gamma)$ of these ovals are three disjoint intervals of the real axis with end-points that are branch points of the hyperelliptic surface $\mathcal{M}_*$ (Fig. 13(a)). Besides α, β, and γ there exists another triple of ovals on $\mathcal{M}_*$, $\widetilde{\alpha}$, $\widetilde{\beta}$, and $\widetilde{\gamma}$, on which the function $y(m)$ takes real values. Each oval in one group intersects precisely two ovals in the other group (Fig. 13(b)).

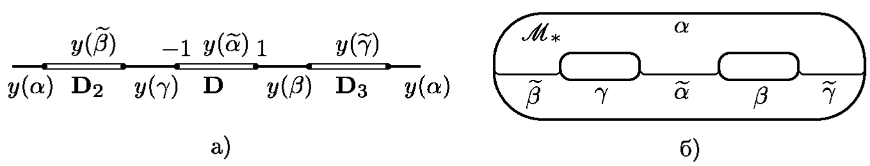


Figure 13. (a) Two-sheeted representation of the surface $\mathcal{M}_*(\lambda, \partial\Gamma)$; (b) six ovals

6.2.3. Projective structure. Associating each point $m \in \mathcal{M}_*$ with the point $p(m)$ in the membrane $\mathcal{F}$ we obtain a single-valued branch of the projective structure p on the surface $\mathcal{M}_*$ cut along two cycles. This structure has two simple branch points, and to calculate its monodromy one can equip the three cycles $C \subset \overline{\mathcal{F}}_1$, $\widehat{\mathbb{R}} \subset \overline{\mathcal{F}}_2$, and Γ on the surface with the linear fractional transformations $\chi(\mathbf{D})$, $\chi(\mathbf{D}_2)$, and $\chi(\mathbf{D}_3)$, respectively. It will be clear from the next subsection that these three cycles can be moved by an isotopy to the three ovals $\widetilde{\alpha}$, $\widetilde{\beta}$, and $\widetilde{\gamma}$, respectively, on which the function $y(m)$ takes real values. Thus, the monodromy of the structure p is induced by the monodromy on the Riemann sphere with six punctures that is defined by the three cuts $y(\widetilde{\alpha})$, $y(\widetilde{\beta})$, and $y(\widetilde{\gamma})$ along the real axis and the matrices $\mathbf{D}$, $\mathbf{D}_2$, and $\mathbf{D}_3$ corresponding to them. It is now easy to find a rational function R_3 of degree 3 such that $\mathcal{M}(R_3) = \mathcal{M}_*(\lambda, \partial\Gamma)$, and the monodromy of p is $\chi \circ \mathbf{T}_{\mathcal{M}}$ (see the last line of the table in Theorem 2.1.1). Up to transformations (24) the function R_3 can be recovered from its critical values, which are the end-points of the intervals $y(\widetilde{\beta})$ and $y(\widetilde{\gamma})$, and its monodromy $\mathbf{T}_*$. By Theorem 6, λ is an eigenvalue of the PS_3 equation with this rational parameter R_3. We have thus proved the following result.

Proposition. *Each* $\lambda \in (1,2)$ *is an eigenvalue of some* (*depending on* λ) PS_3 *integral equation.*

It is easy to understand what must be modified to obtain representations for other eigenvalue-eigenvector pairs of this PS_3 equation. The construction of the membrane $\mathcal{F}$ must be changed. For instance, using the 'grafting' procedure [11], [12], [15], when one stitches an annulus onto the membrane, one can modify the projective structure keeping its monodromy, but changing the underlying complex structure of $\mathcal{M}$. To return to the original complex structure on $\mathcal{M}$ one must change three parameters of the complex structure on the surface $\mathcal{M}_*$ defined by the membrane. One of them is the spectral parameter λ. It could be useful to list all projective structures (constructions of membranes) yielding 'eigenpairs' of the PS_3 equation, in the spirit of [15].

6.2.4. Membrane with additional symmetry. The solution of the Riemann monodromy problem recovered from the projective structure in the previous subsection ranges in general on a non-degenerate quadric. Condition (53) for odd loops $[\mathbf{r}]$ means that the ramification points of $\mathcal{F}$ are taken into one another by the second-order rotation $\chi(\mathbf{D}_3)$. In this subsection we consider a two-parameter family of such membranes.

Let Γ be a cut that is not merely symmetric relative to the unit circle, but also lies on the straight line $\varepsilon^2\widehat{\mathbb{R}}$. In this case we can explicitly describe the hyperelliptic involution H of the surface $\mathcal{M}_*$. Namely, the point p in one of the sheets $\mathcal{F}_1$, $\mathcal{F}_2$ is taken to the point $\chi(\mathbf{D}_3)p$ in the other sheet. The consistency of this definition can easily be verified by considering the cycles $C \subset \overline{\mathcal{F}}_1$, $\widehat{\mathbb{R}} \subset \overline{\mathcal{F}}_2$, and Γ. The involution H has 6 fixed points a, b, c, d, e, f, which are plotted on the sheet $\mathcal{F}_2$ on the left-hand side of Fig. 14.

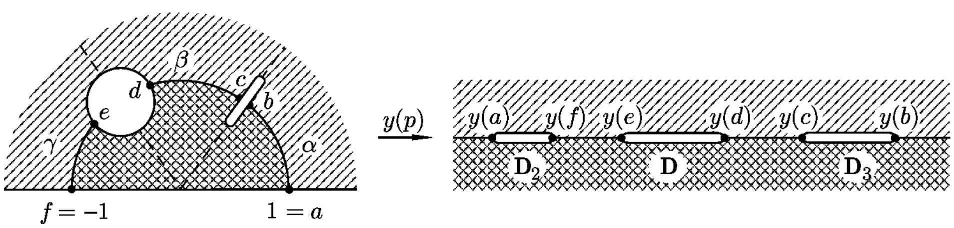


Figure 14. Representation of function $y(p)$ for symmetric membrane $\mathcal{F}$

We now point out an explicit representation for the function $y(m)$ in (62) and for the branch points of the two-sheeted surface $\mathcal{M}_*$. The second-order element y is invariant under the hyperelliptic involution, therefore it takes real values on the cycles $C = \widetilde{\alpha}$, $\widehat{\mathbb{R}} = \widetilde{\beta}$, and $\Gamma = \widetilde{\gamma}$ in the surface. Thus, the function $y(m)$ performs a conformal map of each of the four domains resulting from cutting the sheets $\mathcal{F}_1$ and $\mathcal{F}_2$ along the unit circle and Γ onto the upper or the lower half-plane, in accordance with the symmetry principle. It is easy to see for the family of surfaces

$\mathcal{M}_*(\lambda, \partial\Gamma)$ under consideration that the corresponding projective structures p have the symmetry (57) for odd loops, so that the solution W of the Riemann monodromy problem ranges on a cone.

§ 7. Conclusion

One comes across various interpretations of the concept of *solution* for equations of mathematical physics. We can distinguish three traditional lines of approach to this question. Those taking the first line believe that a problem is solved once the *existence* of a solution in some function class is established, which means that the singularities of the solution are specified and its smoothness and integrability properties are determined. A physicist will not be satisfied because each problem that is properly posed is a reflection of some physical reality whose existence he has never put in question.

Another tradition is that of the *numerical* solution of problems: for instance, in engineering. The importance of this approach lies in its connections with applications of the mathematical science to practice. Still, a numerical solution does not seem to be satisfactory to the intellect because it is difficult to take a discrete collection of numbers for a solution of a problem involving the continuum. It can be said with hindsight that the numerical approach is not adequate to the nature of the solution.

The third line of tradition consists in finding *explicit representations* for solutions: for instance, formulae. Such a solution must be considered the most valuable one because from a formula one can see the existence of a solution, understand its global and local properties, and can use it for numerical computations. Of course, there can also be disappointing results on the non-existence of some or other representation. For instance, the general polynomial equation cannot be solved in radicals, general ordinary differential equations are not soluble in quadratures, and so on. In connection with such problems Poincaré wrote: "Après de longs et vains efforts pour les ramener à des problèmes plus simples, les géomètres se sont enfin résignés à les étudier pour eux-mêmes, et ils ont été récompensés par le succès". In other words, one must seek new forms of constructive representation of solutions. According to Poincaré, associated with each equation is some family of transcendental functions the analysis of whose properties (including methods of their calculation) enables one to understand the underlying physical phenomenon. This was the guiding idea of our work.

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