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Connection between the existence of a priori estimate for a flux and the convergence of iterative methods for diffusion equation with highly varying coefficients

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Abstract — An iterative method with the number of iterations independent of the coefficient jumps is proposed for the boundary value problem for a diffusion equation with highly varying coefficient. The method applies one solution of the Poisson equation at each step of iteration. In the present paper we extend the class of domains the iterative method is justified for.

Keywords: Elliptic equations, iterative methods, rate of convergence independent of the coefficient jump

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In [1] (see also [2] and the references therein) an iterative method with the convergence rate not dependent on the coefficient jump was proposed to solve the system of linear algebraic equations obtained in approximation of the diffusion equation with a highly varying piecewise constant coefficient.

The proof of convergence of the iterative method is based on the proof of the existence of an a priori estimate in L_2 for the flow $k\nabla u$ with a constant independent of the jump in the diffusion coefficient. At the same time, restrictions on the domain and the diffusion coefficient were of the following nature. It was assumed that the domain is divided into a finite number of disjoint subdomains and the diffusion coefficient k is a piecewise constant highly varying function. In this case, the subdomain with the large value of k should be surrounded by subdomains with $k = 1$. In the present paper, we extend the class of domains for which it was possible to prove the convergence of the iterative method with the rate independent of the jump in the diffusion coefficient.

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1. Formulation of the problem and a priori estimate

For definiteness sake, we consider the Dirichlet boundary value problem for the diffusion equation in the domain $\bar{\Omega} = \bigcup_{i=1}^{3} \bar{\Omega}_i$, where

$$\begin{aligned} \Omega &= \{x = (x_1, x_2),\ 0 < x_i < 1,\ i = 1, 2\} \\ \Omega_1 &= \{x = (x_1, x_2),\ 0 < x_1 < 1,\ 0 < x_2 < 0.5\} \\ \Omega_2 &= \{x = (x_1, x_2),\ 0 < x_1 < 0.5,\ 0.5 < x_2 < 1\} \\ \Omega_3 &= \{x = (x_1, x_2),\ 0.5 < x_1 < 1,\ 0.5 < x_2 < 1\} \end{aligned}$$

the diffusion coefficient $k \geqslant 2$ is piecewise constant and highly varying under the passage from one subdomain to another one.

We consider the boundary value problem

$$-\mathrm{div}\,(k\nabla u) = -\mathrm{div}\,((1+\omega)\nabla u) = f, \quad u|_{\partial\Omega} = 0. \tag{1.1}$$

Here

$$\omega(x) = \begin{cases} 1, & x \in \Omega_1 \\ a_i = k_i - 1 \gg 1, & x \in \Omega_i,\ i = 2, 3. \end{cases}$$

Here and below the equations are understood in the weak sense. Namely, equation (1.1) is equivalent to the equality

$$(k\nabla u, \nabla v) = (f, v) \quad \forall v \in H_0^1(\Omega). \tag{1.2}$$

The letter c with subscripts or without them denotes, as usual, some constants independent of the functions entering the inequalities.

We get an a priori estimate for the flow vector function in the norm of the space $\mathbf{L}_2$. Assuming $v = u$ in (1.2), we obtain

$$\left\|\sqrt{k}\nabla u\right\| \leqslant \|f\|_{-1} \equiv \sup_{v \in H_0^1} \frac{(f, v)}{\|\nabla v\|}.$$

Let us show that in fact we have a more strong estimate in this case. Namely, the following assertion is valid.

Theorem 1.1. *The weak solution to problem* (1.1) *satisfies the a priori estimate*

$$\|k\nabla u\| \leqslant c\|f\|_{-1} \tag{1.3}$$

where c does not depend on a_i.

Proof. Denote $\|v\|_D^2 = (v, v)_D = \int_D v^2 \mathrm{d}x$. For definiteness sake, let $\|k\nabla u\|_{\Omega_3} \geqslant \|k\nabla u\|_{\Omega_2} \geqslant \|k\nabla u\|_{\Omega_l}$. Extend u from Ω_3 to Ω preserving the class and norm H^1

so that the extended function $\widetilde{u}$ should vanish on the boundary Ω (in the sense of traces). In this case, (1.2) implies

$$k_3\|\nabla u\|^2_{\Omega_3} = -(k\nabla u, \nabla\widetilde{u})_{\Omega_1\cup\Omega_3} + (f,\widetilde{u}) = -k_2(\nabla u, \nabla\widetilde{u})_{\Omega_2} - 2(\nabla u, \nabla\widetilde{u})_{\Omega_1} + (f,\widetilde{u})$$
$$\leqslant c_1\left(k_2\|\nabla u\|_{\Omega_2}\|\nabla u\|_{\Omega_3} + \|\nabla u\|_{\Omega_1}\|\nabla u\|_{\Omega_3} + \|f\|_{-1}\|\nabla u\|_{\Omega_3}\right)$$

or

$$k_3\|\nabla u\|_{\Omega_3} \leqslant c_1\left(k_2\|\nabla u\|_{\Omega_2} + \|\nabla u\|_{\Omega_1} + \|f\|_{-1}\right). \tag{1.4}$$

If $k_2\|\nabla u\|_{\Omega_2} \leqslant \|\nabla u\|_{\Omega_1} + \|f\|_{-1}$, then we get the required estimate. Otherwise, (1.4) implies

$$k_3\|\nabla u\|_{\Omega_3} \leqslant 2c_1k_2\|\nabla u\|_{\Omega_2}. \tag{1.5}$$

Consider the latter case. Extend the weak solution u to problem (1.1) from $\Omega_2\cup\Omega_3$ to Ω preserving the class H^1_0 and the norm. As before, denote the extended function by $\widetilde{u}$. Assuming $v = \widetilde{u}$ in (1.2), we get

$$k_2\|\nabla u\|^2_{\Omega_2} + k_3\|\nabla u\|^2_{\Omega_3} = -2(\nabla u, \nabla\widetilde{u})_{\Omega_1} + (f,\widetilde{u}).$$

This equality and (1.5) imply the inequality

$$k_3\|\nabla u\|_{\Omega_3}\left(\|\nabla u\|_{\Omega_2} + \|\nabla u\|_{\Omega_3}\right) \leqslant c_2\left(\|\nabla u\|_{\Omega_1}\|\nabla u\|_{\Omega_2\cup\Omega_3} + \|f\|_{-1}\|\nabla u\|_{\Omega_2\cup\Omega_3}\right).$$

Using the elementary inequality $\sqrt{a^2+b^2} \leqslant a+b,\ \ a,b \geqslant 0$, we obtain the final estimate

$$k_3\|\nabla u\|_{\Omega_3} \leqslant c_3\left(\|\nabla u\|_{\Omega} + \|f\|_{-1}\right)$$

which implies the required assertion.

Note that the proof of [1] does not work in this case because the considerations of [1] required that the subdomain with the greater coefficient k should be surrounded by subdomains with the coefficient k of order one. This requirement does not hold in this case.

Corollary 1.1. The technique of proof implies that estimate (1.3) will be valid for the case when the domain Ω is a union of disjoint subdomains $\Omega_i,\ i = 0,\dots,n$. In this case, $k = 1$ in Ω_0 and $k = \omega_i \gg 1$ in $\Omega_i,\ i = 1,\dots,n$, and a function can be extended from any union of subdomains to the entire domain preserving the class and norm.

Corollary 1.2. Let the partitioning of the domain satisfy the hypothesis of Corollary 1.1. The coefficient k is said to be *variable and highly varying* if it has the following representation $k(x) = h(x)g(x)$, where $h(x),\ 0 < h_1 \leqslant h(x) \leqslant h_2$ is a bounded piecewise smooth function and g is a highly varying piecewise constant function (such as k above). Estimate (1.3) is valid in this case as well.

2. Iterative process

Below we consider the case $k(x) = h(x)(1+\omega)$, where $h(x)$, $0 < h_1 \leqslant h(x) \leqslant h_2$ is a bounded piecewise smooth function and ω has been defined above. In order to construct an iterative solver for (1.1), write down the boundary value problem as a saddle point operator, i.e.,

$$\begin{aligned} -\Delta u + \operatorname{div}(h\mathbf{p}) &= f \\ \alpha\mathbf{p} + \nabla u = 0, &\quad \alpha = \frac{1}{\omega}. \end{aligned} \tag{2.1}$$

Recall that the solution to problem (2.1) is understood in the weak sense, i.e.,

$$\begin{aligned} (\nabla u, \nabla v) - (h\mathbf{p}, \nabla v) = (f, v) \quad \forall v \in H_0^1(\Omega) \\ (\alpha\mathbf{p}, \mathbf{q}) + (\nabla u, \mathbf{q}) = 0 \quad \forall \mathbf{q} \in \mathbf{L}_2(\Omega). \end{aligned} \tag{2.2}$$

We construct the iterative process for problem (2.1). To do that, write down the following two-layer *completely implicit* iterative process:

$$\begin{aligned} Bv_t - \Delta\widehat{v} + \operatorname{div}(h\widehat{\mathbf{q}}) &= f \\ \beta\tau q_t + \alpha\widehat{\mathbf{q}} + \nabla\widehat{v} = 0, &\quad \alpha = \frac{1}{\omega}. \end{aligned} \tag{2.3}$$

The operator B is supposed to be symmetric and positive definite. Formulation (2.3) assumes usual notations accepted in the theory of difference schemes, i.e.,

$$v = v^n, \quad \widehat{v} = v^{n+1}, \quad v_t = \frac{(\widehat{v} - v)}{\tau}.$$

Here τ and β are iterative parameters.

The initial conditions are formed by the functions $v^0 \in H_0^1$, $\mathbf{q}^0 = \nabla h$, $h \in H_0^1$. In particular, we may take $v^0 = 0$, $\mathbf{q}^0 = \mathbf{0}$.

The second equation of (2.3) is explicitly solvable relative to $\widehat{\mathbf{q}}$,

$$\widehat{\mathbf{q}} = \frac{\beta}{\beta+\alpha}\mathbf{q} - \frac{1}{\beta+\alpha}\nabla\widehat{v}. \tag{2.4}$$

Substituting this expression into the first equation of (2.3) and taking into account the equalities $\widehat{v} = v + \tau v_t$, $\widehat{\mathbf{q}}_t = \mathbf{q} + \tau\mathbf{q}_t$, we get

$$\left(B - \tau\Delta - \operatorname{div}\left(\frac{h\tau}{\beta+\alpha}\nabla\right)\right)v_t = \Delta v - \operatorname{div}\left(\frac{h\beta}{\beta+\alpha}\mathbf{q}\right) + \operatorname{div}\left(\frac{h}{\beta+\alpha}\nabla v\right) + f.$$

Thus, the implementation of iterative process (2.3) requires the equation with the operator

$$C = B - \tau\Delta - \operatorname{div}\left(\frac{h\tau}{\beta+\alpha}\nabla\right)$$

to be ‘easily solvable’. In this case, given known values of v and $\mathbf{q}$, we can calculate $\widehat{v}$ by solving the equation

$$Cv_t = \Delta v - \operatorname{div}\left(\frac{h\beta}{\beta+\alpha}\mathbf{q}\right) + \operatorname{div}\left(\frac{h}{\beta+\alpha}\nabla v\right) + f \tag{2.5}$$

after that, $\widehat{\mathbf{q}}$ is obtained from explicit formula (2.4).

For definiteness sake, assume that we have an efficient solution algorithm for the Dirichlet problem for the Poisson equation in the whole domain. In this case, the algorithm requires one solution of the Dirichlet problem for the Poisson equation in the whole domain Ω at each step of the iterative process in the differential case. In the discrete case, we may take for C any easily invertible symmetric positive definite operator (for example, the identity operator). The rate of convergence for sufficiently small τ will not depend on the jump of k, but, generally speaking, it depends on the discretization parameter.

The proof of convergence of the iterative process basically coincides with [1]. Let $C = -\Delta$. In this case we have

$$B = -\Delta + \tau\Delta + \operatorname{div}\left(\frac{h\tau}{\beta+\alpha}\nabla\right).$$

The following condition is sufficient for the operator B to be positive definite:

$$\tau\Big(1 + h_2/\beta\Big) < 1 \tag{2.6}$$

where $h_2 = \max h$.

Take some $\beta > 0$ and fix $\tau > 0$ so that (2.6) holds true. In this case, there exist positive constants γ_1 and γ_2 such that the following condition is valid:

$$-\gamma_1\Delta \leqslant B \leqslant -\gamma_2\Delta. \tag{2.7}$$

Write down equation for the error $w = v - u$, $\mathbf{r} = \mathbf{q} - \mathbf{p}$:

$$\begin{aligned} &Bw_t - \Delta\widehat{w} + \operatorname{div}(h\widehat{\mathbf{r}}) = 0 \\ &\beta\tau\mathbf{r}_t + \alpha\widehat{\mathbf{r}} + \nabla\widehat{w} = \mathbf{0}, \quad \alpha = \frac{1}{\omega}. \end{aligned} \tag{2.8}$$

Denote

$$\|\mathbf{r}\|_h^2 = (h\mathbf{r},\mathbf{r}), \quad \|v\|_{\Omega_i} = \int_{\Omega_i} v^2 \mathrm{d}x, \quad \|\nabla v\| = \|v\|_1.$$

Take the scalar products of the first equation and $2\tau\widehat{w}$ in L_2 and of the second equation and $2h\tau\widehat{\mathbf{r}}$ in $\mathbf{L}_2$. Adding the results, we get

$$\|\widehat{w}\|_B^2 - \|w\|_B^2 + \tau^2\|w_t\|_B^2 + 2\tau\|\widehat{w}\|_1^2 + \beta\tau\|\widehat{\mathbf{r}}\|_h^2 - \beta\tau\|\mathbf{r}\|_h^2 + \beta\tau^3\|\mathbf{r}_t\|_h^2 + 2\tau(\alpha h\widehat{\mathbf{r}},\widehat{\mathbf{r}}) = 0$$

which implies the estimate

$$\|\widehat{w}\|_B^2 - \|w\|_B^2 + \tau^2\|w_t\|_B^2 + 2\tau\|\widehat{w}\|_1^2 + \beta\tau\|\widehat{\mathbf{r}}\|_h^2 - \beta\tau\|\mathbf{r}\|_h^2 + 2\tau(h\widehat{\mathbf{r}},\widehat{\mathbf{r}})_{\Omega_1} \leqslant 0. \quad (2.9)$$

Estimate the norm $\|\widehat{\mathbf{r}}\|$ from the first equation of (2.8). Note that the choice of initial conditions and (2.4) imply that $\widehat{\mathbf{r}}$ has the form $\widehat{\mathbf{r}} = d(x)\nabla g$, where d is a piecewise constant function. By d_i we denote the value of d in Ω_i. For definiteness sake, let $\|\widehat{\mathbf{r}}\|_{\Omega_3} \geqslant \|\widehat{\mathbf{r}}\|_{\Omega_2} \geqslant \|\widehat{\mathbf{r}}\|_{\Omega_1}$. Extend g from Ω_3 to Ω so that $\widetilde{g} \in H_0^1(\Omega)$ and $\|\nabla\widetilde{g}\| \leqslant c\|\nabla g\|_{\Omega_3}$. Taking the scalar product of the first equation of (2.8) and $\widetilde{g}$ in L_2, we get

$$\begin{aligned} h_1\|\widehat{\mathbf{r}}\|_{\Omega_3}^2 &\leqslant (h\widehat{\mathbf{r}},\widehat{\mathbf{r}})_{\Omega_3} = (Bw_t,\widetilde{g}) + (\nabla\widehat{w},\nabla\widetilde{g}) - (h\widehat{\mathbf{r}},\nabla\widetilde{g})_{\Omega_2} - (h\widehat{\mathbf{r}},\nabla\widetilde{g})_{\Omega_1} \\ &\leqslant c\Big(\|w_t\|_B\,\|\widetilde{g}\|_B + \|\widehat{w}\|_1\,\|\widetilde{g}\|_1 + h_2\|\widehat{\mathbf{r}}\|_{\Omega_2}\,\|\nabla\widetilde{g}\|_{\Omega_2}) + h_2\|\widehat{\mathbf{r}}\|_{\Omega_1}\,\|\nabla\widetilde{g}\|_{\Omega_1}\Big) \\ &\leqslant c_2\Big(\|w_t\|_B + \|\widehat{w}\|_1 + \|\widehat{\mathbf{r}}\|_{\Omega_2} + \|\widehat{\mathbf{r}}\|_{\Omega_1}\Big)\|\widehat{\mathbf{r}}\|_{\Omega_3}. \end{aligned}$$

Dividing by $\|\widehat{\mathbf{r}}\|_{\Omega_3}$, we get

$$\|\widehat{\mathbf{r}}\|_{\Omega_3} \leqslant c_3\Big(\|w_t\|_B + \|\widehat{w}\|_1 + \|\widehat{\mathbf{r}}\|_{\Omega_2} + \|\widehat{\mathbf{r}}\|_{\Omega_1}\Big). \quad (2.10)$$

Let us consider two cases. If $\|\widehat{\mathbf{r}}\|_{\Omega_2} \leqslant \|w_t\|_B + \|\widehat{w}\|_1 + \|\widehat{\mathbf{r}}\|_{\Omega_1}$, then the previous relation implies the inequality

$$\|\widehat{\mathbf{r}}\|_{\Omega_3} \leqslant 2c_3\Big(\|w_t\|_B + \|\widehat{w}\|_1 + \|\widehat{\mathbf{r}}\|_{\Omega_1}\Big)$$

which gives

$$\|\widehat{\mathbf{r}}\|_{\Omega_2\cup\Omega_3}^2 \leqslant c_4\Big(\|w_t\|_B^2 + \|\widehat{w}\|_1^2 + \|\widehat{\mathbf{r}}\|_{\Omega_1}^2\Big). \quad (2.11)$$

If $\|\widehat{\mathbf{r}}\|_{\Omega_2} > \|w_t\|_B + \|\widehat{w}\|_1 + \|\widehat{\mathbf{r}}\|_{\Omega_1}$, then (2.10) implies the inequality

$$\|\widehat{\mathbf{r}}\|_{\Omega_3} \leqslant 2c_3\|\widehat{\mathbf{r}}\|_{\Omega_2}. \quad (2.12)$$

In this case we estimate $\|\widehat{\mathbf{r}}\|$ in the same way as in the proof of the a priori estimate. Namely, extend g from $\Omega_2\cup\Omega_3$ to Ω preserving the class and norm. Let $\widetilde{g}$ be the extended function. Take the scalar product of the first equation of (2.8) and g. We have

$$\begin{aligned} h_1d_2\|\nabla g\|_{\Omega_2}^2 + h_1d_3\|\nabla g\|_{\Omega_3}^2 &\leqslant (hd\nabla g,\nabla g)_{\Omega_2\cup\Omega_3}^2 \\ &= (Bw_t,\widetilde{g}) + (\nabla\widehat{w},\nabla g) + (hd\nabla g,\nabla\widetilde{g})_{\Omega_1} \\ &\leqslant \|w_t\|_B\|\widetilde{g}\|_B + \|\widehat{w}\|_1\,\|\widetilde{g}\|_1 + h_2\|\widehat{\mathbf{r}}\|_{\Omega_1}\,\|\widetilde{g}\|_{\Omega_1} \\ &\leqslant c\Big(\|w_t\|_B + \|\widehat{w}\|_1 + \|\widehat{\mathbf{r}}\|_{\Omega_1}\Big)\Big(\|\nabla g\|_{\Omega_2} + \|\nabla g\|_{\Omega_3}\Big). \end{aligned}$$

Estimate from above the left-hand side of the latter inequality with the use of (2.12). We have

$$\begin{aligned} h_1 d_2 \|\nabla g\|_{\Omega_2}^2 + h_1 d_3 \|\nabla g\|_{\Omega_3}^2 &= h_1 \Big(\|\widehat{\mathbf{r}}\|_{\Omega_3} \|\nabla g\|_{\Omega_3} + \|\widehat{\mathbf{r}}\|_{\Omega_2} \|\nabla g\|_{\Omega_2} \Big) \\ &\geqslant h_1 \Big(\|\widehat{\mathbf{r}}\|_{\Omega_2} \|\nabla g\|_{\Omega_2} + \|\widehat{\mathbf{r}}\|_{\Omega_3} \|\nabla g\|_{\Omega_3} \Big) \\ &\geqslant h_1 \|\widehat{\mathbf{r}}\|_{\Omega_3} \left(\|\nabla g\|_{\Omega_3} + \frac{1}{2c_3} \|\nabla g\|_{\Omega_2} \right) \\ &\geqslant c \|\widehat{\mathbf{r}}\|_{\Omega_3} \Big(\|\nabla g\|_{\Omega_3} + \|\nabla g\|_{\Omega_2} \Big). \end{aligned}$$

The last two inequalities give an estimate of form (2.11) (possibly, with another constant). Thus, we have proved that the norm $\|\widehat{\mathbf{r}}\|_{\Omega_2 \cup \Omega_3}$ satisfies estimate (2.11) in any case.

Multiply both sides of (2.11) by $\gamma\tau^2$, where $\gamma > 0$ will be determined further. Adding the obtained result to (2.9), we get

$$\begin{aligned} &\|\widehat{w}\|_B^2 - \|w\|_B^2 + \tau^2 (1 - c_4 \gamma) \|w_t\|_B^2 + \tau (2 - c_4 \gamma \tau) \|\widehat{w}\|_1^2 \\ &\quad + \beta\tau \|\widehat{\mathbf{r}}\|_h^2 - \beta\tau \|\mathbf{r}\|_h^2 + \tau (2h_1 - c_4 \gamma \tau) \|\widehat{\mathbf{r}}\|_{\Omega_1}^2 + \gamma\tau^2 \|\widehat{\mathbf{r}}\|_{\Omega_2 \cup \Omega_3}^2 \leqslant 0. \end{aligned} \tag{2.13}$$

Take γ so that the following inequalities hold:

$$c_4 \gamma \leqslant 1, \quad c_4 \gamma \tau \leqslant h_1.$$

In this case, taking into account (2.13), the previous inequalities, and the boundedness of h from above and below, we obtain the final inequality

$$\|\widehat{w}\|_B^2 + \beta\tau \|\widehat{\mathbf{r}}\|_h^2 \leqslant \Big(1 + c_5 \tau \Big)^{-1} \Big(\|w\|_B^2 + \beta\tau \|\mathbf{r}\|_h^2 \Big). \tag{2.14}$$

The constant c_5 in (2.14) does not depend on the jump of the coefficient k. Thus, we have proved the following assertion.

Theorem 2.1. *Let the parameters β and τ of the iterative process be such that the operator B is positive definite. In this case, iterative method* (2.5)*,* (2.4) *converges with the rate of geometric progression with the exponent not dependent on the jump of the coefficient k.*

It is not difficult to see that Theorem 2.1 is valid for domains satisfying the conditions of Corollary 1.1.

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