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Optimal stochastic forcings for sensitivity analysis of linear dynamical systems

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Abstract: The paper is devoted to the construction of optimal stochastic forcings for studying the sensitivity of linear dynamical systems to external perturbations. The optimal forcings are sought to maximize the Schatten norms of the response. As an example, we consider the problem of constructing the optimal stochastic forcing for the linear dynamical system arising from the analysis of large-scale structures in a stratified turbulent Couette flow.

Keywords: Linear dynamical systems, stochastic forcing, Lyapunov equations, singular values, Schatten norms.

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One approach to studying the sensitivity of a dynamical system being in a steady state is to analyze its response to stochastic forcing in this state. Stochastic forcing is used to simulate various external influences, which are not taken into account within the initial system. If small perturbations are of interest, it is sufficient to study the system linearized with respect to the steady state under consideration. In particular, this approach was used to study laminar [2, 9, 10] and turbulent [14, 15] hydrodynamic flows. For example, the analysis performed in [14, 15] highlighted the characteristic spatial scales and forms of the large-scale organized structures observed in turbulent flows against a background of small-scale turbulence (see [8] and its bibliography).

In the above-mentioned papers, the stochastic forcing was a delta-correlated Gaussian stochastic process with zero mean and uniform spectral density. However, the simplifying assumption of uniform spectral density might not be adequate, for example, when stochastic forcing simulates generation of small-scale turbulence because its intensity might be anisotropic and depend on spatial coordinates. This paper focuses on the construction of the optimal stochastic forcing, which is a delta-correlated Gaussian stochastic process with zero mean and, in general, non-uniform spectral density. The optimization problem is posed by analogy with the linear optimal control problems [23], with the solution of that problem being found in the Schatten norms.

The structure of the paper is as follows. In Section 1, the problem of constructing the optimal stochastic forcing is posed and solved. Section 2 demonstrates the results of applying the proposed approach to construction of the optimal stochastic forcing for a linear dynamical system used in [22] to analyze the large-scale organized structures in a stratified turbulent Couette flow. Section 3 summarizes this work.

Throughout this paper, $\|\cdot\|_2$ denotes the 2-norm for vectors and matrices, the identity matrices are denoted by I and their size will be clear from the context, T denotes the symbol of transposition, and '$*$' denotes the symbol of conjugate transposition.

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1 Stochastic forcing of the linear dynamical systems

We further refer to the inhomogeneous system of linear ordinary differential equations of the form

$$\frac{\mathrm{d}u}{\mathrm{d}t} = Au + Vf \tag{1.1}$$

as the linear dynamical system, where t is a time, $u = u(t)$ is a vector-valued function, A is a given square complex matrix of order n whose eigenvalues have negative real parts, f is, in general, a time-dependent m-component column ($m \leqslant n$), which we further call the forcing function, and V is a matrix of size $n \times m$. Note that Vf lies in the linear span of the columns of V. In particular, this allows to consider the cases where only a part of the system variables is forced, choosing a corresponding matrix V. If the forcing of a general form is of interest, then V should be the identity matrix.

Assume that we are interested in the following response of system (1.1) to the forcing:

$$g = W^* u \tag{1.2}$$

where W is an $n \times r$ matrix, $n \geqslant r$. Further $g(t)$ will be called the response function. If the forcing is intended to influence all components of the solution u, then W should be the identity matrix. Otherwise, when, for example, we want to maximize the influence on some solution components, W should be the rectangular matrix formed of the corresponding columns of the identity matrix of order n.

1.1 Response to stochastic forcing

Let the forcing function $f(t)$ in (1.1), where $-\infty < t < \infty$, be a delta-correlated Gaussian stochastic process with zero mean and spectral density C:

$$\langle f(t) \rangle = 0, \quad \langle f(t) f(t')^* \rangle = C\delta(t - t'), \quad C = C^* \geqslant 0$$

where $\delta(\tau)$ is the Dirac delta-function, and $\langle \cdot \rangle$ is the ensemble averaging. Then the response function g defined in (1.2) will be a Gaussian stochastic process with zero mean and a time-independent covariance matrix (see [20]):

$$\langle g(t) g(t)^* \rangle = \mathcal{M}(C) = W^* X W \tag{1.3}$$

where X is the solution of the Lyapunov equation

$$AX + XA^* = -\widetilde{C} \tag{1.4}$$

with $\widetilde{C} = VCV^*$. Since all eigenvalues of A have negative real parts, equation (1.4) has a unique solution for any $\widetilde{C}$ [12] that can be found using the efficient matrix algorithms (see, e.g., [3, 13]).

The eigenvectors of the covariance matrix $\widetilde{X} = \mathcal{M}(C)$ form an orthonormal basis, which is referred to as the basis of empirical orthogonal functions (EOF). Decomposition with respect to this basis is also known as Principal Component Analysis (PCA, see, e.g., [19]), Proper Orthogonal Decomposition (POD, see, e.g., [21]) and Karhunen–Loève decomposition (see, e.g., [10]). The first EOF, i.e. the eigenvector of $\widetilde{X}$ corresponding to its largest eigenvalue $\lambda_{\max}(\widetilde{X})$, contributes most to the variance maintained by the stochastic forcing, being the main configuration observed in the response time series of the stochastically forced system (1.1). The share of total variance contributed to the first EOF is $s(C) = \lambda_{\max}(\widetilde{X})/\mathrm{tr}(\widetilde{X})$, where $\mathrm{tr}(\widetilde{X})$ is the trace of $\widetilde{X}$.

Let us notice one important particular case. Let $f(t) = c\varphi(t)$, where c is a time-independent deterministic vector and $\varphi(t)$ is a scalar delta-correlated Gaussian stochastic process with zero mean and unit variance. Then, similar to (1.3), the following formula for the trace of covariance matrix of response function is valid:

$$\mathrm{tr}\langle g(t) g(t)^* \rangle = \langle \|g(t)\|_2^2 \rangle = c^* \widetilde{Y} c$$

where

$$\widetilde{Y} = V^* Y V \tag{1.5}$$

with Y being the solution of the Lyapunov equation

$$A^*Y + YA = -WW^*$$

whose operator is adjoint to that of equation (1.4) in Frobenius inner product (see [5, p. 92]).

If we choose as c the eigenvector of $\widetilde{Y}$ corresponding to its largest eigenvalue, then the corresponding forcing function maximizes the mean square of the 2-norm of the response function on the set of forcings of the form $f(t) = c\varphi(t)$, where vector c has the same 2-norm as this eigenvector.

1.2 Optimal spectral density

For an $n \times n$ matrix B, the quantity

$$|||B|||_p = \left(\sum_{j=1}^{n} \sigma_j(B)^p \right)^{1/p}$$

is referred to as the Schatten p-norm [5], where $\sigma_1(B) \geqslant \ldots \geqslant \sigma_n(B)$ denote algebraically complete set of singular values of B. Note that $|||B|||_2 = \|B\|_F$ is the Frobenius norm, and $|||B|||_\infty = \|B\|_2$ is the spectral norm. If B is a Hermitian nonnegative definite matrix, then $|||B|||_1 = \operatorname{tr}(B)$ is the trace of B. In addition, we will use, without further explanation, the following statement: let B' and B'' be Hermitian matrices of the same size, with B'' being a nonnegative definite matrix, and $-B'' \leqslant B' \leqslant B''$. Then $|||B'|||_p \leqslant |||B''|||_p$ for $p = 1, 2$, and ∞.

We consider the problem of constructing the stochastic forcing that gives the maximum response in the Schatten p-norms at $p = 1, 2$, and ∞. This problem is posed as the optimization problem for spectral density C:

$$C \neq 0 : \quad \frac{|||\mathcal{M}(C)|||_p}{|||C|||_p} \to \max, \quad C = C^* \geqslant 0 \tag{1.6}$$

where $\mathcal{M}$ denotes the mapping defined in (1.3).

Theorem 1.1. *Let all eigenvalues of A have negative real parts. Then the solution of problem* (1.6) *at $p = 1$ is the matrix $C = cc^*$ of rank 1, where c is the eigenvector of matrix* (1.5) *corresponding to its largest eigenvalue; at $p = 2$ solution is the right singular vector of mapping* (1.3) *corresponding to its largest singular value; and at $p = \infty$ solution is the identity matrix of order m.*

Proof. Note that, under the hypothesis of Theorem 1.1, mapping (1.3) has the following property: $\mathcal{M}(C') \leqslant \mathcal{M}(C'')$ for any Hermitian matrices satisfying $C' \leqslant C''$. This inequality clearly follows from the similar property of the Lyapunov equations [12].

Let us start with the case $p = \infty$. Let C be a non-zero Hermitian nonnegative definite matrix. Then, first of all,

$$|||C|||_\infty = \lambda_{\max}(C)|||I|||_\infty$$

where $\lambda_{\max}(C)$ is the largest eigenvalue of C. Second,

$$C \leqslant \lambda_{\max}(C)I \implies \mathcal{M}(C) \leqslant \lambda_{\max}(C)\mathcal{M}(I)$$

and therefore,

$$|||\mathcal{M}(C)|||_\infty \leqslant \lambda_{\max}(C)|||\mathcal{M}(I)|||_\infty.$$

Thus,

$$\frac{|||\mathcal{M}(C)|||_\infty}{|||C|||_\infty} \leqslant \frac{|||\mathcal{M}(I)|||_\infty}{|||I|||_\infty}.$$

Consider now the case $p = 1$. Let us represent the matrix C as a sum of Hermitian nonnegative definite matrices of rank 1 (for example, based on the spectral decomposition):

$$C = \sum_{j=1}^{\nu} C_j$$

where v is the rank of C. Denote by C' a matrix which solves the optimization problem (1.6) at $p = 1$ on the set of Hermitian nonnegative definite matrices of rank 1. Let us introduce the notation $\alpha = \alpha_1 + \ldots + \alpha_v$ where $\alpha_j = |||C_j|||_1 / |||C'|||_1$.

It is not difficult to see that $|||C|||_1 = \alpha |||C'|||_1$ and

$$|||\mathcal{M}(C)|||_1 = ||| \sum_{j=1}^{v} \mathcal{M}(C_j) |||_1 = \sum_{j=1}^{v} |||\mathcal{M}(C_j)|||_1 \leqslant \sum_{j=1}^{v} |||\mathcal{M}(\alpha_j C')|||_1 = \alpha |||\mathcal{M}(C')|||_1 .$$

Thus,

$$\frac{|||\mathcal{M}(C)|||_1}{|||C|||_1} \leqslant \frac{|||\mathcal{M}(C')|||_1}{|||C'|||_1}.$$

It remains to take into account that, as shown in Section 1.1, the optimal spectral density of rank 1 at $p = 1$ is the matrix cc^*, where c is the eigenvector of matrix (1.5) corresponding to its largest eigenvalue.

Consider now the case $p = 2$. Using the Lyapunov equation (1.4), the linear mapping (1.3) can be represented in a matrix-vector form as $\tilde{x} = M\tilde{c}$. Here $\tilde{x}$ and $\tilde{c}$ are r^2-component and m^2-component columns, respectively, obtained by stacking the columns of $\tilde{X}$ and C, and M is the $r^2 \times m^2$ matrix of the linear mapping (1.3) having the following form:

$$M = \left(W^T \otimes W^*\right) \left(I \otimes A + \overline{A} \otimes I\right)^{-1} \left(\overline{V} \otimes V\right) \tag{1.7}$$

where $\otimes$ denotes the Kronecker product, and bar denotes the element-wise complex conjugation. Since

$$|||\tilde{X}|||_2 = \|\tilde{x}\|_2, \qquad |||C|||_2 = \|\tilde{c}\|_2$$

problem (1.6) at $p = 2$ is reduced to the following matrix optimization problem

$$\tilde{c} \neq 0 : \ \frac{\|M\tilde{c}\|_2}{\|\tilde{c}\|_2} \to \max \tag{1.8}$$

on the set of m^2-component columns obtained by stacking the columns of non-zero $m \times m$ Hermitian nonnegative matrices. If problem (1.8) is considered on the set of all nonzero m^2-component columns, then its solution is the right singular vector of M corresponding to its largest singular value. The corresponding matrix C (obtained by reshaping the column $\tilde{c}$) is the right singular vector of mapping (1.3) corresponding to its largest singular value. Thus, to complete the proof, it is sufficient to show that the right singular vector of mapping (1.3) corresponding to its largest singular value can be chosen Hermitian nonnegative definite. This statement is a consequence of a more general statement that is of interest on its own. We state and prove it below. □

Theorem 1.2. *Let all eigenvalues of A have negative real parts. Then, the right singular vector of mapping* (1.3) *corresponding to an arbitrary singular value can be chosen Hermitian, and that corresponding to the largest singular value can be chosen Hermitian nonnegative definite.*

Proof. The mapping adjoint to (1.3) can be represented as

$$\mathcal{M}^*(\tilde{X}) = V^* Z V \tag{1.9}$$

with Z being the solution of the Lyapunov equation:

$$A^* Z + ZA = -W\tilde{X}W^* . \tag{1.10}$$

Formula (1.9) can be derived, based on the conjugate transpose of the matrix M (1.7). The right singular vector C of mapping (1.3) corresponding to its singular value σ is the eigenvector of $\mathcal{M}^*(\mathcal{M}(\cdot))$ corresponding to its eigenvalue $\lambda = \sigma^2$:

$$\mathcal{M}^*(\mathcal{M}(C)) = \lambda C. \tag{1.11}$$

If C is a skew-Hermitian matrix, then the matrix C multiplied by the imaginary unit is the sought Hermitian singular vector. Otherwise, taking into account the equality:

$$\mathcal{M}^*(\mathcal{M}(C^*)) = \lambda C^*$$

which is directly obtained from (1.11) by the conjugate transposition, the matrix $C + C^*$ is the sought Hermitian singular vector.

Let now C be an arbitrary Hermitian matrix of order m, with $C = Q\Lambda Q^*$ being its spectral decomposition, where Q is a unitary matrix of eigenvectors, and Λ is a diagonal matrix of eigenvalues. Having replaced the diagonal elements of Λ by their absolute values, we denote the result as Λ' and consider the matrix $C' = Q\Lambda' Q^*$. Note that, first of all, $|||C|||_2 = |||C'|||_2$, and second, $-C' \leqslant C \leqslant C'$, and therefore,

$$-\mathcal{M}(C') \leqslant \mathcal{M}(C) \leqslant \mathcal{M}(C') \implies |||\mathcal{M}(C)|||_2 \leqslant |||\mathcal{M}(C')|||_2.$$

If C is the right singular vector of mapping (1.3) corresponding to its largest singular value, then the latter inequality becomes an equality. Therefore, it clearly follows that the constructed non-zero Hermitian non-negative definite matrix C' is the right singular vector of mapping (1.3) corresponding to its largest singular value. □

Note that in [7] it was proved that the singular vector corresponding to the smallest singular value of the real Lyapunov operator, whose spectrum lies in the left half-plane, can be chosen symmetric. It is equivalent to the same statement on the right singular vector corresponding to the largest singular value of the inverse real Lyapunov operator. Theorem 1.2 considers a much more general complex operator (if V and W are identity matrices, it is the inverse complex Lyapunov operator) and proves that this singular vector can be chosen Hermitian nonnegative definite. Thus, this theorem generalizes and extends the results of [7].

The statements similar to Theorem 1.1, when V and W are identity matrices, were proved in [16, 17]. Subsequently it was shown in [4, 6] that these statements follow from more general theory on the norms of linear positive operators, to which the mapping $\mathcal{M}$ (1.3) belongs in this case. The proposed proof of Theorem 1.1 is original and based only on well-known results of the matrix analysis.

2 Numerical experiments

In this section, we discuss the results of numerical experiments with a linear dynamical system proposed in [22] to analyze the large-scale organized structures emerging against a background of small-scale turbulence in a stratified turbulent Couette flow [11]. This system is obtained by parameterizing the turbulent stresses with the isotropic eddy viscosity and diffusivity operators and subsequent linearization with respect to the mean flow (see [22] for more details). The disturbances harmonic in the horizontal spatial directions are of interest. Approximating the equations for the amplitudes of such disturbances in the wall-normal direction and projecting the result onto the subspace of the non-divergent grid functions [18], we obtain a linear system of the form (1.1) with zero matrix V. The matrix A of this system depends on the Reynolds number Re, the Richardson number Ri, and the streamwise α and spanwise γ disturbance wavenumbers (see [22] for details).

We consider matrices A corresponding to $\mathrm{Re} = 4 \cdot 10^4$, $\mathrm{Ri} = 0.03$ and two pairs of the wavenumber values: $\alpha = 0.39$, $\gamma = 1.16$, at which the maximum amplification of the disturbance energy is reached [22], and $\alpha = 0$, $\gamma = 0.25$, at which the response to the stochastic forcing in all considered Schatten norms is close to the maximum. Figure 1 shows the leading part of the spectrum of A in these two cases. It can be seen that in the former case, the leading eigenvalues are much more stretched along the imaginary axis, but they are much farther away from it.

To study the sensitivity, forcings of the form Vf should be added to the right-hand side of the described homogeneous system and various observation matrices W should be used. Since this paper is devoted first to the computational technology and its justification, we present further numerical experiments only with the identity matrices V and W.

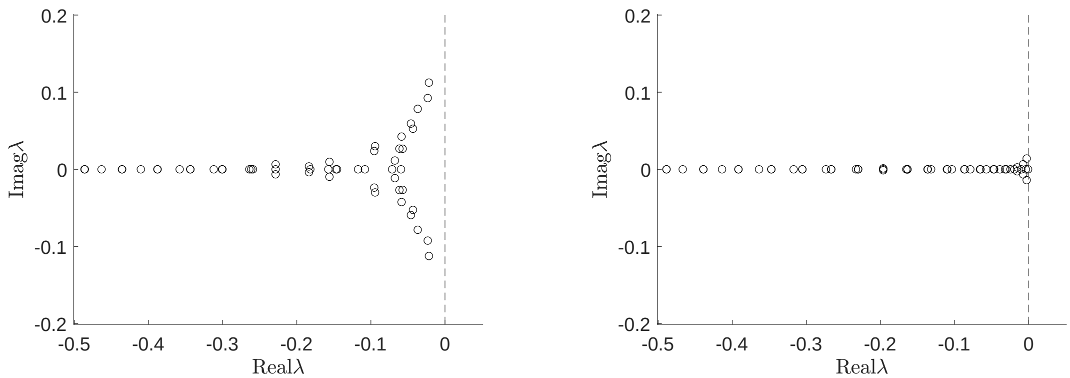

Fig. 1: The leading part of the spectrum of A at the wavenumber values $\alpha = 0.39$, $\gamma = 1.16$ (left) and $\alpha = 0$, $\gamma = 0.25$ (right).

p	$\rho_1(C_p)$	$\rho_2(C_p)$	$\rho_\infty(C_p)$	$s(C_p)$
1	1366	1012	970	0.71
2	1042	1109	1135	0.67
∞	10	99	1485	0.46

Tab. 1: The response to optimal stochastic forcings in different p-norm and the share of the total variance corresponding to the first EOF at $\alpha = 0.39$, $\gamma = 1.16$.

p	$\rho_1(C_p)$	$\rho_2(C_p)$	$\rho_\infty(C_p)$	$s(C_p)$
1	8082	7886	7885	0.97
2	7296	7936	7984	0.97
∞	35	514	8872	0.84

Tab. 2: The response to optimal stochastic forcings in different p-norm and the share of the total variance corresponding to the first EOF at $\alpha = 0$, $\gamma = 0.25$.

2.1 Computation of the optimal spectral densities

According to Theorem 1.1, the optimal spectral density of stochastic forcing at the case $p = 1$ is the matrix $C = cc^*$ of rank 1, where c is the eigenvector of (1.5) corresponding to its largest eigenvalue, that at the case $p = \infty$ is the identity matrix of order m, and that at the case $p = 2$ is the right singular vector of the mapping (1.3) corresponding to its largest singular value.

Using Theorem 1.2, the computation of the optimal spectral density for the case $p = 2$ can be done as follows. First we compute the eigenvector of M^*M corresponding to its largest eigenvalue, where M is the matrix (1.7) of the linear mapping (1.3). To compute this eigenvector, any iterative method for solving the partial Hermitian eigenvalue problem, that does not require to form matrix explicitly but uses only a matrix-vector multiplication procedure can be used (e.g., the Lanczos method, the simultaneous iteration, or the conjugate gradient method, see [1] for details). Note that the multiplication of matrices M and M^* by a vector is equivalent to the solution of the corresponding Lyapunov equations (1.4) and (1.10), respectively. After the desired eigenvector is found, we reshape it to a matrix and then convert this matrix to a Hermitian non-negative definite matrix by the algorithm described in the proof of Theorem 1.2.

Let us denote by C_p the optimal spectral density in the Schatten p-norm, and introduce the relative value of the Schatten q-norm of the covariance matrix $\mathcal{M}(C)$ of response function for a given matrix C:

$$\rho_q(C) = \frac{|||\mathcal{M}(C)|||_q}{|||C|||_q}.$$

For brevity, this quantity is further referred to as the response.

For two considered matrices A, Tables 1 and 2 provide the values of responses to the stochastic forcings having the spectral densities optimal in different Schatten p-norms. In the last columns, the values of the share of total response function variance corresponding to the first EOF are given. Since C_p is the optimal spectral density in the p-norm, maximum values across the columns are in the first, second, and third positions, respectively. The optimal spectral densities C_1 and C_2 result in a small value of response in ∞-norm

(1–2 orders of magnitude smaller than the response to C_∞), and the optimal spectral density C_∞ leads to a significantly smaller response in the 1-norm than C_1.

Tables 1 and 2 show that the optimal spectral densities C_1, C_2, and C_∞ lead to significantly different covariance matrices of the response function. For example, the values of the share of the total response function variance corresponding to the first EOF are noticeably larger at spectral densities C_1, C_2 than at C_∞.

3 Conclusions

In the present paper, to study the sensitivity of linear dynamical systems to external forcings, the computational technology is proposed and justified for constructing the optimal spectral density of stochastic forcing, which is a delta-correlated Gaussian stochastic process with zero mean. The optimization problem is posed by analogy to the linear optimal control problems and solved in the Schatten p-norms at $p = 1, 2$, and ∞. As an example of two linear dynamical systems arising in studying of the large-scale organized structures in a stratified Couette turbulent flow, it is shown that the optimal spectral densities in different norms can differ significantly from each other. This leads to the significantly different covariance matrices of the response function. The proposed formulation of the problem and the method for computing the optimal stochastic forcing can be used in the study of the sensitivity of any nonlinear dynamical system being in a steady state.

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