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Low-rank canonical approximations with sequential dimensionality increments

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Introduction

In the field of wireless network technologies, an important role is played by the tensor

$$T_{j_1\dots j_d} = \sum_{k=1}^{R} \gamma_r e^{\boldsymbol{i\phi_k} J},\ T \in \mathbb{C}^{I_1\times\dots\times I_d}$$

where $\phi_1, \dots, \phi_d \in [0, 2\pi]$, $J = j_1(I_2 \dots I_d) + j_2(I_3 \dots I_d) + \dots + j_{d-1}I_d + j_d$.
The main task is to find $\phi_1, \dots, \phi_d$.

Canonical poliadic decomposition

Let's consider the CPD (Canonical poliadic decomposition) of T. Denote the obtained factor matrices as $A_1, \dots, A_d$. They will have the following structure

$$A_j = \begin{bmatrix} 1 & \dots & 1 \\ e^{i\phi_1(I_1\cdot\ldots\cdot I_{j-1})} & \dots & e^{i\phi_R(I_1\cdot\ldots\cdot I_{j-1})} \\ \dots & \dots & \dots \\ e^{i\phi_1(I_1\cdot\ldots\cdot I_{j-1})(I_j-1)} & \dots & e^{i\phi_R(I_1\cdot\ldots\cdot I_{j-1})(I_j-1)} \end{bmatrix}$$

It is also known from the article [1] that CPD has a noise suppression property. Thus, we can reduce the problem of finding $\phi_1, \dots, \phi_d$ to the problem of finding CPD. One of the most popular ways to find it is the ALS.

1 Extrapolation

Let $\widetilde{T} \in \mathbb{C}^{l_1\times l_2\times\ldots\times l_d}, l_1 \leq I_1, \dots, l_d \leq I_d$ be a subtensor of tensor $T + W$, obtained by selecting elements with indices $q_1^{(1)}, \dots, q_{l_1}^{(1)}$ along the first coordinate, $q_1^{(2)}, \dots, q_{l_2}^{(2)}$ along the second, and so on. The first columns of factor matrices of $\hat{T}$ will have the form

$$\widetilde{a}_1^{(k)} \approx \hat{C} \begin{bmatrix} e^{i\psi_k q_1} \\ e^{i\psi_k q_2} \\ \vdots \\ e^{i\psi_k q_{l_n}} \end{bmatrix}$$

Since $\widetilde{T}$ is a subtensor of $T + W$, it must hold that $\psi_k \approx \phi_k$. Thus, using the columns of the factor matrices of the subtensor $\widetilde{T}$ we can obtain the decomposition of the tensor T.

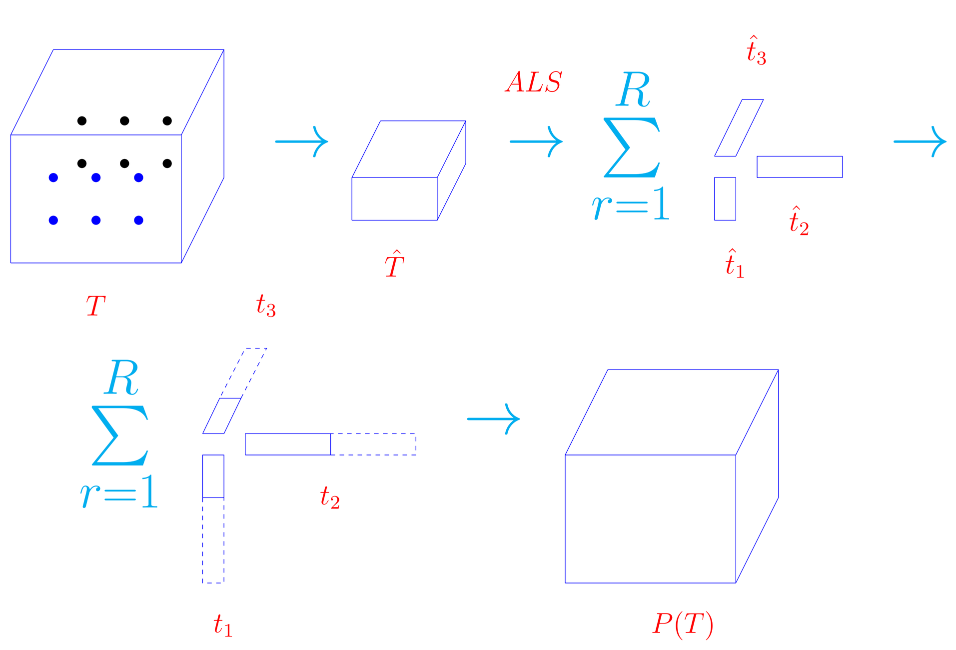


Figure 1: Extrapolation

2 Dimensionality increasing

Consider the inequality from [1]

$$\|\mathcal{P}(\mathbf{A}+\mathbf{W}) - \mathbf{A}\|_F \lesssim \sqrt{\frac{dM^{\frac{1}{d}}\log(M)}{M}}\mathbb{E}\|\mathbf{W}\|_F$$

and note that as the dimensionality of the tensor d increases, the right-hand side will decrease. Therefore, the idea arises: to artificially increase the order of the tensor as much as possible to obtain the best approximation accuracy. However, this idea has two problems: increasing of the computational complexity and significantly worsens ALS convergence. Solution: increase the dimensionality gradually through the decomposition of factor matrices.

Let us describe the process of decomposing the factor matrix $A_1 \in \mathbb{C}^{I_1\times R}$ (the other matrices are decomposed similarly). Let I_1 be a composite number and $I_1 = n\cdot m$. Take its first column a_1 and form a matrix $B \in \mathbb{C}^{n\times m}$ from it

$$a_1 = \begin{bmatrix} 1 \\ e^{i\phi_1} \\ \dots \\ e^{i\phi_1(mn-1)} \end{bmatrix} \Rightarrow B = \begin{bmatrix} 1 & e^{i\phi_1\cdot n} & \dots & e^{i\phi_1\cdot(m-1)n} \\ e^{i\phi_1} & e^{i\phi_1\cdot(n+1)} & \dots & e^{i\phi_1\cdot((m-1)n+1)} \\ \dots & \dots & \dots & \dots \\ e^{i\phi_1\cdot(n-1)} & e^{i\phi_1\cdot(2n-1)} & \dots & e^{i\phi_1\cdot(mn-1)} \end{bmatrix}$$

Matrix B is one-rank matrix and can be represented

$$B = \begin{bmatrix} 1 \\ e^{i\phi_1} \\ \dots \\ e^{i\phi_1\cdot(n-1)} \end{bmatrix} \begin{bmatrix} 1 & e^{i\phi_1\cdot n} & \dots & e^{i\phi_1\cdot(m-1)n} \end{bmatrix}$$

We will perform similar actions with the remaining columns of A_1 and collect the corresponding vectors into matrices $\hat{A}_1 \in \mathbb{C}^{n\times R}$ and $\hat{A}_2 \in \mathbb{C}^{m\times R}$. Thus, from the tensor $T \in \mathbb{C}^{I_1\times I_2\times\ldots\times I_d}$, we obtain an element-wise matching tensor $T_1 \in \mathbb{C}^{n\times m\times\ldots\times I_d}$, whose order is $d+1$. After decomposing $A_1, \dots, A_d$ we'll get factor matrices $\hat{A}_1, \dots, \hat{A}_{\hat{d}}$ of tensor $\hat{T} \in \mathbb{C}^{\hat{I}_1\times\ldots\times\hat{I}_{\hat{d}}}$ with order $\hat{d} \geq d$. After this we use some iterations of ALS on $\hat{T}$ with initial matrices $\hat{A}_1, \dots, \hat{A}_{\hat{d}}$ and repeat our actions.

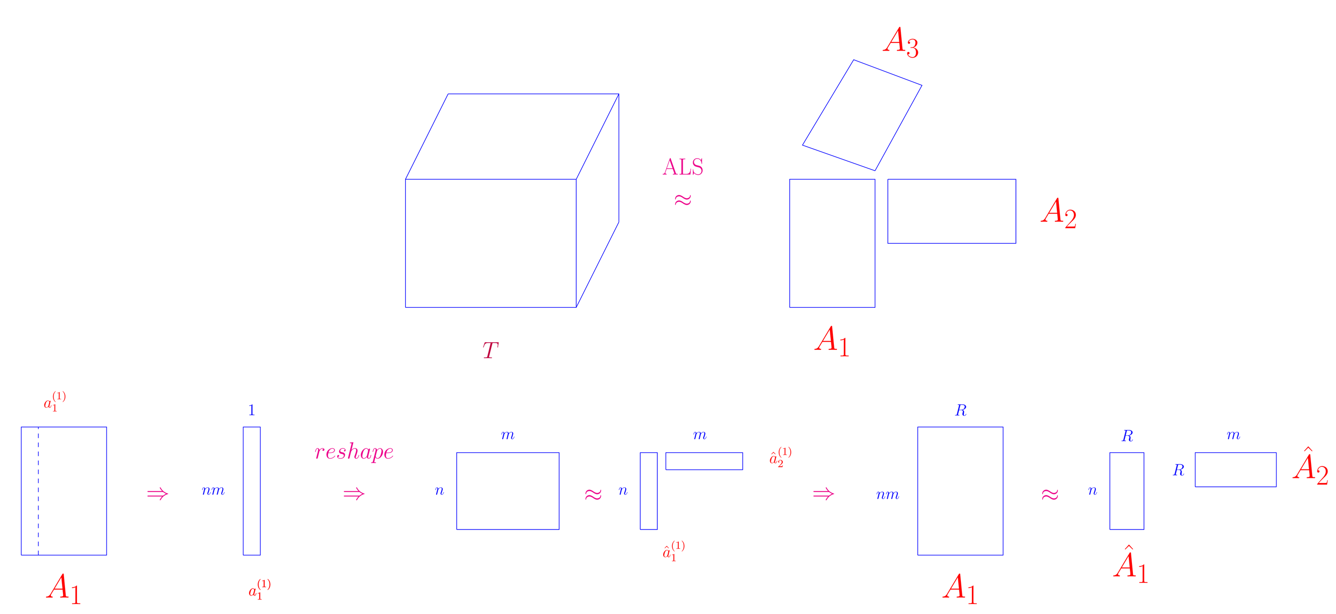
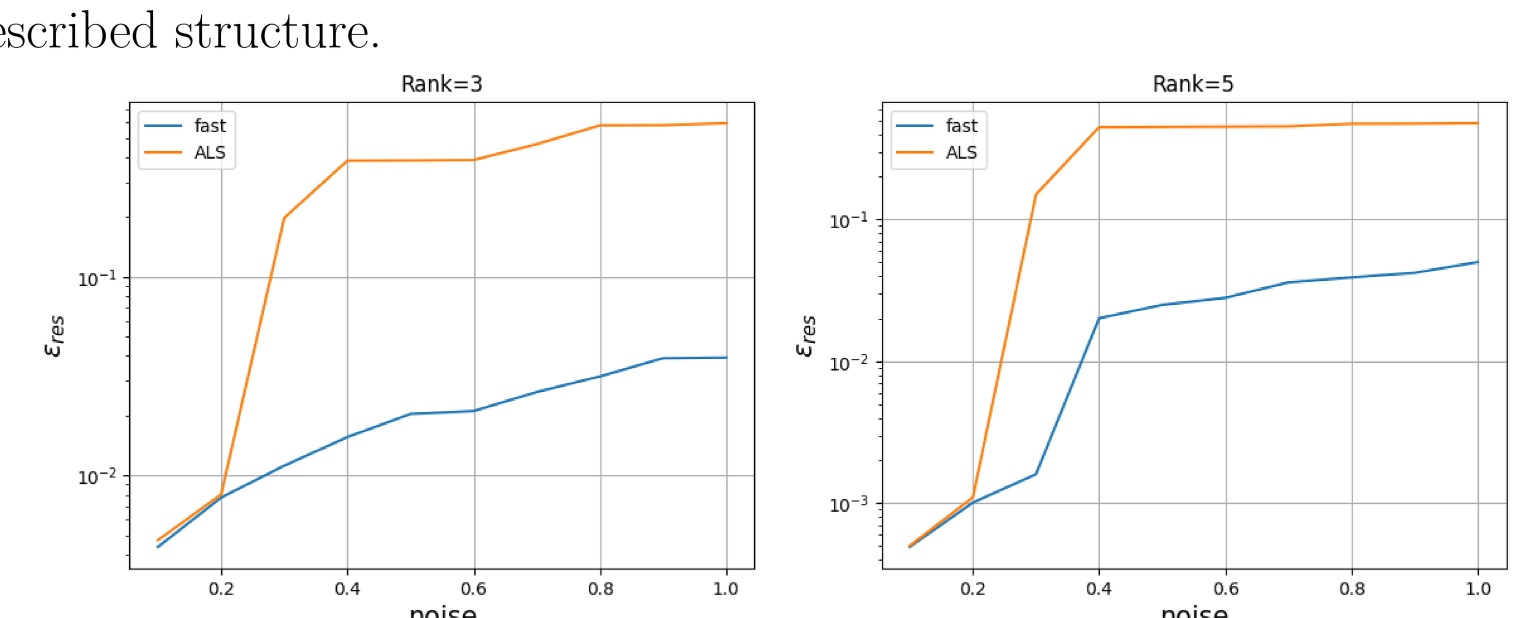


Figure 2: Factor matrix decomposition

3 Numerical experiments

We will conduct experiments on the tensor $T \in \mathbb{C}^{32\times32\times32}$ with the previously described structure.



References

[1] Nikolai Zamarashkin Sergey Petrov. Theoretical bounds for noise filtration using low-rank tensor approximations.