THE FIRST SCHOOL-CONFERENCE ON TENSOR METHODS IN MATHEMATICS AND DATA SCIENCE
10-21 November 2024, Shenzhen

On the quantized TT-ranks of moment sequences

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QUANTIZED TENSOR TRAIN FORMAT

The tensor train (TT for short) decomposition is a memory-efficient representation of tensors [4] with the use of the following idea

$$A(i_1, i_2, \dots, i_{d-1}, i_d) =$$

$$\sum_{\alpha_1=1}^{R_1} \dots \sum_{\alpha_{d-1}}^{R_{d-1}} G_1(i_1, \alpha_1)\, G_2(\alpha_1, i_2, \alpha_2) \dots G_{d-1}(\alpha_{d-2}, i_{d-1}, \alpha_{d-2})\, G_d(\alpha_{d-1}, i_d).$$

The numbers $R_1, \dots, R_d$ are called the *TT-ranks* of A.

This idea can be applied to vectors via the suitable "reshaping". For a vector v of the size 2^d we denote by $Q(v)$ the tensor with size $2 \times \dots \times 2 = 2^d$ obtained from v by the formula

$$Q(d, f)(i_1, \dots, i_d) = v(2^{d-1} i_1 + 2^{d-2} i_2 + \dots i_d + 1), \quad i_1, \dots, i_d \in \{0, 1\}.$$

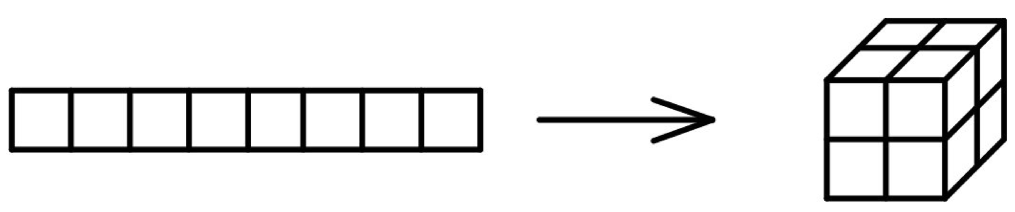

Figure 1: Illustration of the QTT format.

The expression of the vector v as the TT decomposition of $Q(v)$ is called [3] the *quantized tensor train format* for v. The corresponding TT-ranks are called the *QTT-ranks* of v.

We consider "functionally generated" vectors, i.e. vectors of the form $f[d] = (f(1), \dots, f(2^d))$, where $f = \{f(k)\}_{k \in \mathbb{N}}$ is a number sequence. We find the sufficient conditions that ensure that the vector $f[d]$ has an approximation v such that $\|f[d] - v\|_2 \le \varepsilon$ and QTT-ranks of v are bounded by R from above. More precisely, under certain conditions (i.e. for moment sequences with an additional boundedness assumption) we prove that there exists such an approximation v with $R = \mathcal{O}(d(\ln d + \ln(1/\varepsilon)))$.

Similar results are known [3] for the ∞-norm and a wider class of sequences. For applications it can be preferable to have an estimation of the 2-norm, since the standard algorithm TTSVD [4] is particularly well-behaved with respect to the Frobenius' norm.

HAMBURGER PROBLEM AND MOMENTS

Definition 1. If μ is a Borel measure on $\mathbb{R}$ such that all polynomials are μ-integrable, then the numbers

$$M_k(\mu) = \int_{\mathbb{R}} t^{k-1} d\mu(t), \quad k \in \mathbb{N}$$

are called the *moments* of μ.

Given a sequence $f(k) \in \mathbb{R}$, $k \in \mathbb{N}$ the *Hamburger problem* asks whether there exists a positive Borel measure μ such that $f(k) = M_k(\mu)$ for all k. We say that the Hamburger problem for f is solvable if such measure μ exists.

Given $N \in \mathbb{N}$ we denote by $H(N, f)$ the following $N \times N$ matrix:

$$H(N, f) = \begin{pmatrix} f(1) & f(2) & \dots & f(N) \\ f(2) & f(3) & \dots & f(N+1) \\ \vdots & \dots & \ddots & \vdots \\ f(N) & f(N+1) & \dots & f(2N-1) \end{pmatrix}.$$

Theorem 1 ([5], Theorem X.4)**.** *The Hamburger problem for f is solvable if and only if $H(N, f)$ is positive semi-definite for all N.*

Definition 2. We say that f is H-bounded, if $\sup_N \|H(N, f)\|_2 < +\infty$.

Theorem 2 ([2])**.** *The sequence f is H-bounded if and only if there exists an essentialy bounded (with respect to Lebesgue measure) 2π-periodic function g on $\mathbb{R}$ such that*

$$f(k) = \frac{1}{2\pi} \int_0^{2\pi} g(t) e^{-ikt} dt, \quad k \in \mathbb{N}.$$

Clearly, if f is summable (i.e. $\sum_k |f(k)| < \infty$), then it is H-bounded.

QTT-RANKS OF MOMENT SEQUENCES

Clearly, the unfolding matrices of $Q(d, f)$ are submatrices in $H(N, f)$ for $N \ge 2^d + 1$. Applying results of [1] on singular values of positive semi-definite Hankel matrices and the technique of TTSVD, we obtain the following

Theorem 3 ([6])**.** *Assume that the Hamburger moment problem for f is solvable and that $M = \sup_N \|H(N, f)\|_2 < +\infty$. Then there exists a constant $C > 0$ independent of f such that for all $\varepsilon \in (0, 1)$ and all $d \in \mathbb{N}$ there exists a tensor T of the size $2 \times \dots \times 2 = 2^d$ with tensor train ranks not exceeding $Cd(\ln d + \ln(1/\varepsilon) + \ln M + 1)$ such that $\|Q(d, f) - T\|_F \le \varepsilon$.*

EXAMPLES AND APPLICATIONS

Example 1. Let $f_\alpha(k) = k^{-\alpha}$ for $\alpha > 0$. It can be proved that the Hamburger moment problem for the sequence f admits a solution

$$d\mu(t) = \chi_{[0,1]}(-\ln t)^{\alpha-1} dt / \Gamma(\alpha),$$

where $\chi_{[0,1]}$ is the characteristic function of $[0, 1] \subset \mathbb{R}$. If $\alpha > 1$, then f_α is summable and Theorem 3 is applicable to f_α.

Example 2. Consider f_α from the previous example with $\alpha = 1$. This sequence is not summable, but it can be shown that $f(k)$ equals the k-th Fourier coefficient of the function $g(t) = \sum_{k=1}^{\infty} (e^{ikt} - e^{-ikt})/k = 2i \sum_{k=1}^{\infty} \sin(kt)/k$, which is essentially bounded. Therefore, Theorem 3 is applicable to f_1.

Example 3. Let $\alpha, \beta > 0$ and consider the sequence $f_{\alpha,\beta}(k) = k^{-\alpha} e^{-\beta k}$. From the solution of the Hamburger problem for f_α from Example 1 it is easy to solve the Hamburger problem for $f_{\alpha,\beta}$. Clearly, $f_{\alpha,\beta}$ is summable and, therefore, Theorem 3 is applicable to $f_{\alpha,\beta}$.

In the following figures we show the numerical experiments for f_α.

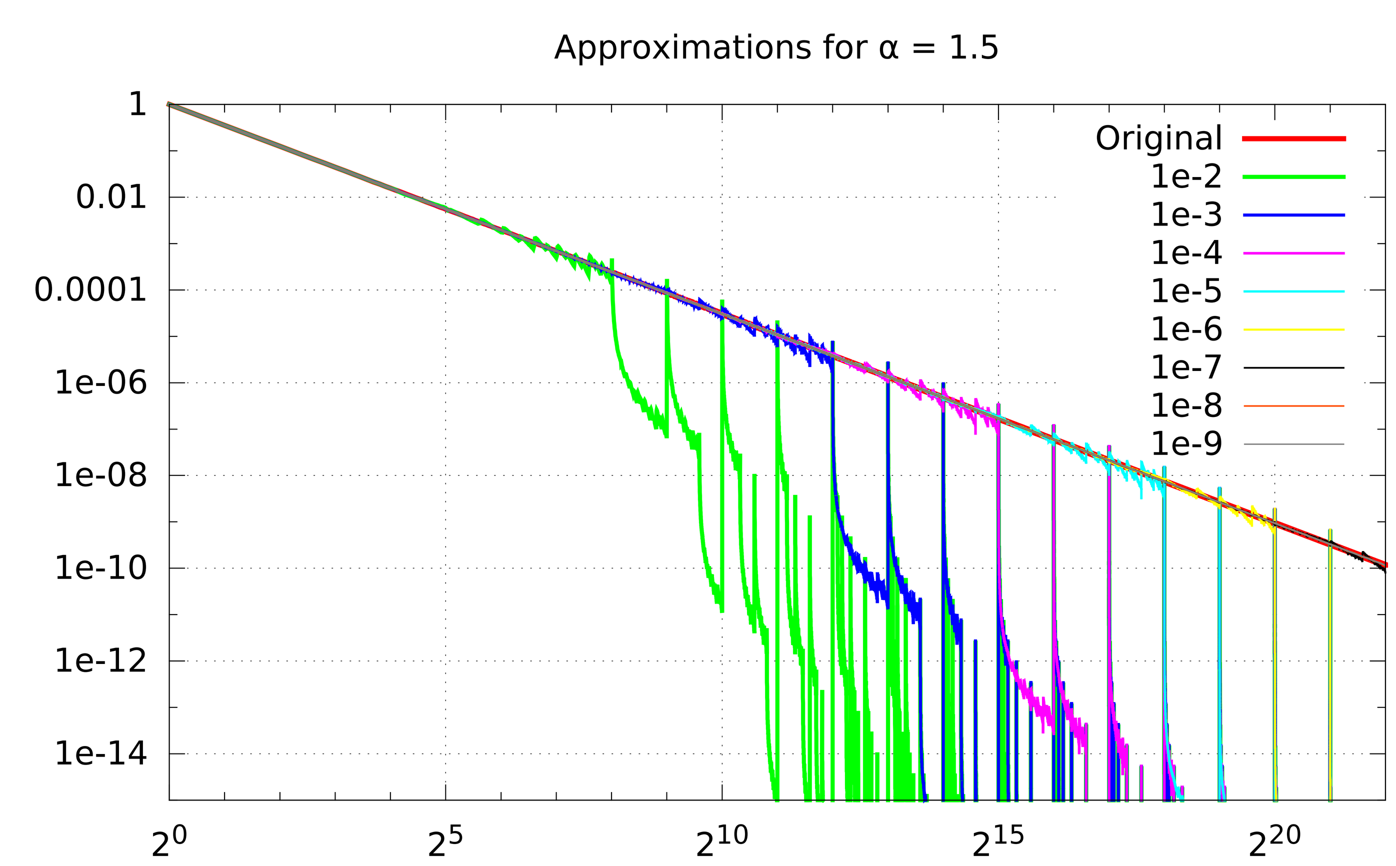

Figure 2: QTT approximations with multiple tolerance levels for $d = 22$ and $\alpha = 1.5$.

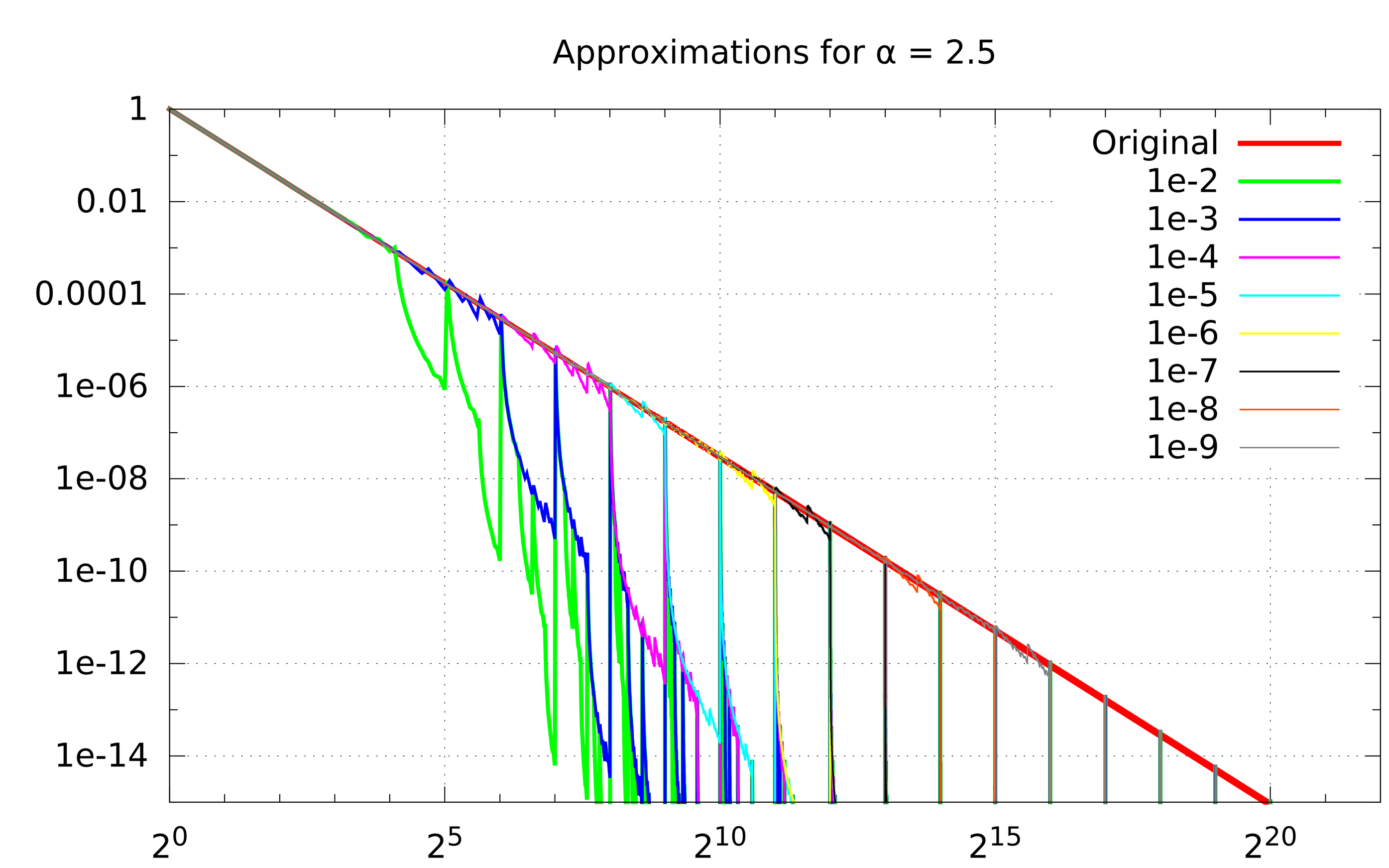

Figure 3: QTT approximations with multiple tolerance levels for $d = 22$ and $\alpha = 2.5$.

[1] Bernhard Beckermann and Alex Townsend. "Bounds on the singular values of matrices with displacement structure". In: *SIAM Rev.* 61.2 (2019), pp. 319–344.
[2] Zeev Nehari. "On bounded bilinear forms". In: *Ann. of Math. (2)* 65 (1957), pp. 153–162. ISSN: 0003-486X.
[3] IV Oseledets. "Approximation of matrices with logarithmic number of parameters". In: *Doklady Mathematics*. Vol. 80. Springer. 2009, pp. 653–654.
[4] Ivan V Oseledets. "Tensor-train decomposition". In: *SIAM Journal on Scientific Computing* 33.5 (2011), pp. 2295–2317.
[5] Michael Reed and Barry Simon. *Methods of modern mathematical physics. II. Fourier analysis, self-adjointness*. Academic Press, New York-London, 1975, pp. xv+361.
[6] MS Smirnov and SA Matveev. "Estimates for the Quantized Tensor Train Ranks for the Power Functions". In: *Lobachevskii Journal of Mathematics* 45.7 (2024), pp. 3182–3187.