On Modifications of the Navier–Stokes Equations

G.M. Kobelkov
Moscow State Univ. & Inst. of Numer. Maths. RAS

Notations

Let Ω be a bounded Lipshitz domain in R^3.

We denote independent variables by $x = (x_1, x_2, x_3)$ or x, y, z if it does not lead to misunderstandings. In the space of vector functions, introduce the norms and operators:

$$\|\mathbf{f}\|^2 = \sum_{i=1}^{2} \int\limits_{\Omega} f_i^2(x)dx = \int\limits_{\Omega} |\mathbf{f}|^2 dx, \quad \|\mathbf{f}_x\|^2 = \sum_{i=1}^{2} \sum_{j=1}^{3} \int\limits_{\Omega} \left(\frac{\partial f_i}{\partial x_j}\right)^2 dx,$$

$$\Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}, \quad \partial_{x_i} = \frac{\partial}{\partial x_i}, \quad \|\cdot\|_q = \|\cdot\|_{L_q},$$

$$\nabla = (\partial_x, \partial_y), \quad \operatorname{div} \mathbf{f} = \partial_x f_1 + \partial_y f_2, \quad |\nabla \mathbf{f}|^2 = \sum_{i,j=1}^{2} \left(\frac{\partial f_i}{\partial x_j}\right)^2,$$

$$|f|_{q,E_x}^q = \int\limits_{-\infty}^{\infty} |f|^q dx, \quad |f|_{q,E_{xy}}^q = \int\limits_{-\infty}^{\infty} \int\limits_{-\infty}^{\infty} |f|^q dxdy.$$

In what follows, we assume summation over repeating indices in products. By c with and without indices we denote constants in inequalities not depending on the functions entering these inequalities but depending in general on initial data of the problem (a form of domain, constants from the embedding theorems, norms of the right-hand sides of equations, time interval, etc.).

Formulation of the problem

The system of Navier–Stokes equations describing dynamics of incompressible viscous flow is of the form

$$
\begin{aligned}
&\mathbf{u}_t - \nu\Delta\mathbf{u} - \nu\partial_z^2\mathbf{u} + \nabla p + (\mathbf{u}\cdot\nabla)\mathbf{u} + w\mathbf{u}_z = \mathbf{f},\\
&w_t - \nu\Delta w - \nu\partial_z^2 w + p_z + (\mathbf{u}\cdot\nabla)w + ww_z = g,\\
&\operatorname{div}\mathbf{u} + w_z = 0,\\
&(\mathbf{u},w)(x,0) = (\mathbf{u}_0,w_0)(x),\ \operatorname{div}\mathbf{u}_0 + \partial_z w_0 = 0,\quad (\mathbf{u},w)\Big|_{\partial\Omega\times[0,T]} = \mathbf{0}.
\end{aligned}
\tag{1}
$$

In practice, there are problems when a viscosity coefficient is different in various directions. For instance, in simulation of ocean dynamics the viscosity coefficients in vertical and horizontal directions are different. So, it is natural to consider the case when the viscosity coefficient in horizontal direction equals ν, while in the vertical direction z it equals μ. In this case equations (1) take the form

$$
\begin{aligned}
&\mathbf{u}_t - \nu\Delta\mathbf{u} - \mu\partial_z^2\mathbf{u} + \nabla p + (\mathbf{u}\cdot\nabla)\mathbf{u} + w\mathbf{u}_z = \mathbf{f},\\
&w_t - \nu\Delta w - \mu\partial_z^2 w + p_z + (\mathbf{u}\cdot\nabla)w + ww_z = g,\\
&\operatorname{div}\mathbf{u} + w_z = 0,\\
&(\mathbf{u},w)(x,0) = (\mathbf{u}^0,w^0)(x),\ \operatorname{div}\mathbf{u}^0 + \partial_z w^0 = 0,\quad (\mathbf{u},w)\Big|_{\partial\Omega\times[0,T]} = \mathbf{0}.
\end{aligned}
\tag{2}
$$

For simplicity of consideration, put $\mathbf{f} = \mathbf{0},\ g = 0$.

Let us study the solvability “in the large” of (2). The following theorem holds:

Theorem 1. *For any sufficiently smooth initial condition $\mathbf{u}_0$, any $\nu > 0$ and arbitrary time interval $[0,T]$ there is $\mu > 0$ such that there exists a solution to (2) “in the large”, i.e. there exists $\mathbf{u} \in \mathbf{H}^1(Q_T)$ satisfying (2) in a week sense and the norm $\|\mathbf{u}_x\|$ is continuous in time on $[0,T]$. Moreover, in this case the following inequality holds*

$$\|\mathbf{u}_t(t)\| \le \|\mathbf{u}_t(0)\| \quad \forall t > 0.$$

Proof. To prove the theorem we use the Ladyzhenskaya inequality

$$\|f\|_4^4 \le c_1 \|f_{x_1}\| \, \|f_{x_2}\| \, \|f_{x_3}\| \, \|f\| \tag{3}$$

being valid for any $f \in H_0^1(\Omega)$.

Take scalar product in $\mathbf{L}_2$ of the first equation of (2) and $\mathbf{u}$ and the second equation of (2) and w in L_2. Adding results, we have

$$\frac{1}{2}\frac{d}{dt}\left(\|\mathbf{u}\|^2 + \|w\|^2\right) + \nu(\|\nabla \mathbf{u}\|^2 + \|\nabla w\|^2) + \mu(\|\mathbf{u}_z\|^2 + \|w_z\|^2) = 0; \tag{4}$$

integration of (4) in time gives

$$\|\mathbf{u}(t)\|^2 + \|w(t)\|^2 \le \|\mathbf{u}^0\|^2 + \|w^0\|^2 \equiv M^2. \tag{5}$$

From (4) and (5) one gets

$$\nu(\|\nabla \mathbf{u}\|^2 + \|\nabla w\|^2) + \mu(\|\mathbf{u}_z\|^2 + \|w_z\|^2) \le M(\|\mathbf{u}_t\| + \|w_t\|). \tag{6}$$

Differentiate (2) in t:

$$\begin{aligned}
&\mathbf{u}_{tt} - \nu\Delta \mathbf{u}_t - \mu\partial_z^2 \mathbf{u}_t + \nabla p_t + (\mathbf{u}_t \cdot \nabla)\mathbf{u} + w_t \mathbf{u}_z + (\mathbf{u} \cdot \nabla)\mathbf{u}_t + w_t \mathbf{u}_{tz} = \mathbf{f}_t, \\
&w_{tt} - \nu\Delta w_t - \mu\partial_z^2 w_t + p_{tz} + (\mathbf{u}_t \cdot \nabla) w + w_t w_z + (\mathbf{u} \cdot \nabla) w_t + w w_{tz} = g_t, \\
&\operatorname{div} \mathbf{u}_t + w_{tz} = 0.
\end{aligned} \tag{7}$$

Take now a scalar product of the first two equations of (7) and $(\mathbf{u}_t, w_t)$:

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(\|\mathbf{u}_t\|^2 + \|w_t\|^2) + \nu(\|\nabla \mathbf{u}_t\|^2 + \|\nabla w_t\|^2) + \mu(\|\mathbf{u}_{tz}\|^2 + \|w_{tz}\|^2) \\
&+((\mathbf{u}_t \cdot \nabla)\mathbf{u} + w_t \mathbf{u}_z, \mathbf{u}_t) + ((\mathbf{u}_t \cdot \nabla) w + w_t w_z, w_t) = 0.
\end{aligned} \tag{8}$$

Estimate the scalar products of (8). Integration by parts gives

$$\begin{aligned}
&|((\mathbf{u}_t \cdot \nabla)\mathbf{u} + w_t \mathbf{u}_z, \mathbf{u}_t) + ((\mathbf{u}_t \cdot \nabla) w + w_t w_z, w_t)| \\
&= |(u_{jt}\mathbf{u}, \partial_{x_j}\mathbf{u}_t) + (w_t \mathbf{u}, \mathbf{u}_{tz}) + (u_{jt} w, \partial_{x_j} w_t) + (w_t w, w_{tz})|.
\end{aligned}$$

Estimate now each of these scalar products separately using the Hölder inequality and estimates (5) and (6):

$$|(u_{jt}\mathbf{u}, \partial_{x_j}\mathbf{u}_t)| \le \|\mathbf{u}_{tx}\| \, \|\mathbf{u}_t\|_4 \|\mathbf{u}\|_4 \le c\|\mathbf{u}_{tx}\|^{7/4}\|\mathbf{u}_t\|^{1/4}\|\nabla\mathbf{u}\|^{1/2}\|\mathbf{u}_z\|^{1/4}$$

$$\le \varepsilon\|\mathbf{u}_{tx}\|^2 + \frac{c}{\varepsilon^7}\|\nabla\mathbf{u}\|^4\|\mathbf{u}_z\|^2\|\mathbf{u}_t\|^2 \le \varepsilon\|\mathbf{u}_{tx}\|^2 + \frac{c}{\varepsilon^7\nu^2\mu}(\|\mathbf{u}_t\|^2 + \|w_t\|^2)^{3/2}.$$

All other scalar products are estimated in the same way. Choosing proper ε, we finally obtain

$$\begin{aligned} &|((\mathbf{u}_t\cdot\nabla)\mathbf{u} + w_t\mathbf{u}_z, \mathbf{u}_t) + ((\mathbf{u}_t\cdot\nabla)w + w_t w_z, w_t)| \\ &\le \frac{\nu}{2}(\|\mathbf{u}_{tx}\|^2 + \|w_{tx}\|^2) + \frac{cM^3}{\nu^9\mu}(\|\mathbf{u}_t^2\| + \|w_t\|^2)^{3/2}. \end{aligned} \tag{9}$$

Substituting (9) into (8), one gets

$$\frac{d}{dt}(\|\mathbf{u}_t\|^2 + \|w_t\|^2) + \nu(\|\nabla\mathbf{u}_t\|^2 + \|\nabla w_t\|^2)$$

$$+\mu(\|\mathbf{u}_{tz}\|^2 + \|w_{tz}\|^2) - \frac{cM^3}{\nu^9\mu}(\|\mathbf{u}_t\|^2 + \|w_t\|^2)^{3/2} \le 0,$$

from what follows

$$\begin{aligned} &\frac{d}{dt}(\|\mathbf{u}_t\|^2 + \|w_t\|^2) + \nu(\|\mathbf{u}_{tx}\|^2 + \|w_{tx}\|^2) \\ &+\left(\mu - \frac{cM^3}{\nu^9\mu}(\|\mathbf{u}_t\| + \|w_t\|)\right)(\|\mathbf{u}_{tz}\|^2 + \|w_{tz}\|^2) \le 0. \end{aligned} \tag{10}$$

It is obvious that the norm $\|\mathbf{u}_t(0)\|$ may be estimated from above by the norm $\|(\mathbf{u}, w)\|_{\mathbf{H}^2}$. Now, from (10) it follows that for any $\nu > 0$ and arbitrary $\|\mathbf{u}_t(0)\| + \|w_t(0)\|$ depending on the norm of initial condition $\|\mathbf{u}_0\|_{\mathbf{H}^2} + \|w_0\|_{H^2}$ there exists $\mu > 0$ such that $\mu - \dfrac{cM^3}{\nu^9\mu}(\|\mathbf{u}_t(0)\| + \|w_t(0)\|) \ge 0$. Then from (10) we conclude that the norm $\|\mathbf{u}_t(t)\|$ satisfies the inequality

$$\|\mathbf{u}_t(t)\| \le \|\mathbf{u}_t(0)\| \quad \forall t > 0. \tag{11}$$

Existence and uniqueness of a solution "in the large" with the help of estimate (11) may be obtained in the same way as in the monograph of O.A.Ladyzhenskaya. The proof is completed.

Improvement of the Ladyzhenskaya modification

Consider now another modification of the Navier-Stokes equations, when the viscosity coefficient is the same in all directions, but the elliptic operator is changed. O.A. Ladyzhenskaya proposed modification of the Navier-Stokes equations allowing to prove existence of a strong solution to (1) "in the large":

$$
\begin{aligned}
&\mathbf{u}_t - \nu\Delta\mathbf{u} - \nu\partial_z^2\mathbf{u} - \nu\varepsilon\left[\mathrm{div}\left(D(\mathbf{u},w)\nabla\mathbf{u}\right) + \partial_z\left(D(\mathbf{u},w)\partial_z\mathbf{u}\right)\right] \\
&\qquad\qquad\qquad + \nabla p + (\mathbf{u}\cdot\nabla)\mathbf{u} + w\mathbf{u}_z = \mathbf{f}, \\
&w_t - \nu\Delta w - \nu\partial_z^2 w - \nu\varepsilon\left[\mathrm{div}\left(D(\mathbf{u},w)\nabla w\right) + \partial_z\left(D(\mathbf{u},w)\partial_z w\right)\right] \\
&\qquad\qquad\qquad + p_z + (\mathbf{u}\cdot\nabla)w + ww_z = g, \\
&\mathrm{div}\,\mathbf{u} + w_z = 0, \\
&(\mathbf{u},w)(x,0) = (\mathbf{u}_0,w_0)(x),\ \mathrm{div}\,\mathbf{u}_0 + \partial_z w_0 = 0, \quad (\mathbf{u},w)\Big|_{\partial\Omega\times[0,T]} = 0,
\end{aligned}
\tag{12}
$$

where

$$
D(\mathbf{u},w) = |\nabla\mathbf{u}|^2 + |\nabla w|^2 + |\partial_z\mathbf{u}|^2 + |\partial_z w|^2. \tag{13}
$$

Consider another modification of the Navier-Stokes equations arising in ocean dynamics. Namely, we consider (12) as modification of (1), but instead of (13) we use $D(\mathbf{u},w) = |\nabla\mathbf{u}|^2$, remove the term $\partial_z\left(D(\mathbf{u},w)\partial_z\mathbf{u}\right)$ from the first equation of (12), and do not change the equation for w. So, we consider the problem

$$
\begin{aligned}
&\mathbf{u}_t - \nu\Delta\mathbf{u} - \nu\partial_z^2\mathbf{u} - \nu\varepsilon\mathrm{div}\left(|\nabla\mathbf{u}|^2\nabla\mathbf{u}\right) \\
&\qquad\qquad\qquad + \nabla p + (\mathbf{u}\cdot\nabla)\mathbf{u} + w\mathbf{u}_z = \mathbf{f}, \\
&w_t - \nu\Delta w - \nu\partial_z^2 w + p_z + (\mathbf{u}\cdot\nabla)w + ww_z = g, \\
&\mathrm{div}\,\mathbf{u} + w_z = 0, \\
&(\mathbf{u},w)(x,0) = (\mathbf{u}_0,w_0)(x),\ \mathrm{div}\,\mathbf{u}_0 + \partial_z w_0 = 0, \quad (\mathbf{u},w)\Big|_{\partial\Omega\times[0,T]} = 0.
\end{aligned}
\tag{14}
$$

To study solvability of (14) "in the large", we need the following lemmas.

Lemma 1. Let $v \in H_0^1[0,l]$. Then the following estimate holds

$$\max_x v^2(x) \le 2\|v_x\|\,\|v\|. \tag{15}$$

Proof. Extend v onto the whole axes by zero. Then

$$v^2(x) = 2\int_{-\infty}^{x} v_x(x)v(x)dx \le 2\|v_x\|\,\|v\|.$$

Q.E.D.

Lemma 2. *Let $f \in H_0^1(\Omega),\ f_x,\ f_y \in L_4(\Omega),\ \Omega \in R^3$, then*

$$\|f\|_5^5 \le \frac{25}{2}\|f_x\|_4\|f_y\|_4\|f_z\|\,\|f\|^2. \tag{16}$$

Proof. Extend by zero the function f onto the whole space R^3. Then

$$|f(x,y,z)|^5 = \frac{25}{4}\int_{-\infty}^{x} \sqrt{|f(x,y,z)|}\, f(x,y,z)\, f_x(x,y,z)dx$$

$$\times \int_{-\infty}^{y} \sqrt{|f(x,y,z)|}\, f(x,y,z)\, f_y(x,y,z)dy.$$

Using the Hölder inequality, from the previous expression we have

$$|f(x,y,z)|^5 \le \frac{25}{4}|f|_{E_x}^{3/2}\,|f_x|_{4,E_x}\,|f|_{E_y}^{3/2}\,|f_y|_{4,E_y}.$$

Integration over R^3 and further implementation of the Hölder inequality give

$$\int_{E_z}\int_{E_{xy}} |f|^5 dxdydz \le \frac{25}{4}\int_{E_z}\left[\int_{E_y} |f|_{E_x}^{3/2}|f_x|_{4,E_x}dy \int_{E_x} |f|_{E_y}^{3/2}|f_y|_{4,E_y}dx\right]dz$$

$$\le \frac{25}{4}\int_{E_z} |f|_{E_{xy}}^3\,|f_x|_{4,E_{xy}}\,|f_y|_{4,E_{xy}}dz \le \frac{25}{4}\max_z |f|_{E_{xy}}^2 \int_{E_z} |f|_{E_{xy}}\,|f_x|_{4,E_{xy}}\,|f_y|_{4,E_{xy}}dz$$

$$\le \ \text{(due to Lemma 1)} \qquad \le \frac{25}{2}\|f_x\|_4\,\|f_y\|_4\,\|f_z\|\,\|f\|^2 dz.$$

Q.E.D.

Corollary. Since $abc \le 0.5a^2b^2 + 0.5c^2 \le 0.25(a^4 + b^4) + 0.5c^2,\ a, b, c \ge 0$, then from (16) it follows

$$\|f\|_5^5 \le \frac{25}{8}(\|f_x\|_4^4 + \|f_y\|_4^4 + 2\|f_z\|^2)\|f\|^2. \tag{17}$$

Let us obtain a proper a priori estimate for a solution to (14). Take a scalar product of (14) and $(\mathbf{u}, w)$:

$$\frac{1}{2}\frac{d}{dt}\left(\|\mathbf{u}\|^2 + \|w\|^2\right) + \nu\left(\|\mathbf{u}_x\|^2 + \|w_x\|^2\right) + \varepsilon\nu\|\nabla\mathbf{u}\|_4^4 = (\mathbf{f}, \mathbf{u}) + (g, w). \tag{18}$$

Obvious estimation of the right-hand side of (18) and further integration in t from 0 to T give

$$\max_{0\le t\le T}(\|\mathbf{u}(t)\|^2 + \|w(t)\|^2) + \nu\int\limits_0^T \left(\|\mathbf{u}_x\|^2 + \|w_x\|^2 + \varepsilon\|\nabla\mathbf{u}\|_4^4\right) dt$$
$$\le c_1\left(\|\mathbf{u}_0\|^2 + \|w_0\|^2 + \frac{1}{\nu}\int\limits_0^T \left(\|\mathbf{f}\|_{-1}^2 + \|g\|_{-1}^2\right) dt\right). \tag{19}$$

Rewrite (18) in another form:

$$\nu\left(\|\mathbf{u}_x\|^2 + \|w_x\|^2\right) + \varepsilon\nu\|\nabla\mathbf{u}\|_4^4 = (\mathbf{f}, \mathbf{u}) + (g, w) - (\mathbf{u}_t, \mathbf{u}) - (w_t, w).$$

Estimating right-hand side and using (19), we get

$$\|\mathbf{u}_x\|^2 + \|w_x\|^2 + \varepsilon\|\nabla\mathbf{u}\|_4^4 \le c_2(\|\mathbf{u}_t\| + \|w_t\| + \|\mathbf{f}\|_{-1}^2 + \|g\|_{-1}^2). \tag{20}$$

Differentiate (14) in t:

$$\begin{aligned}
&\mathbf{u}_{tt} - \nu\mathrm{div}\,((1 + \varepsilon|\nabla\mathbf{u}|^2)\nabla\mathbf{u}_t) - \nu\varepsilon\mathrm{div}\,([|\nabla\mathbf{u}|^2]_t\nabla\mathbf{u}) - \nu\partial_z^2\mathbf{u}_t \\
&\qquad + \nabla p_t + (\mathbf{u}\cdot\nabla)\mathbf{u}_t + (\mathbf{u}_t\cdot\nabla)\mathbf{u} + w\mathbf{u}_{tz} + w_t\mathbf{u}_z = \mathbf{f}_t, \\
&w_{tt} - \nu\Delta w_t - \nu\partial_z^2 w_t + p_{tz} + (\mathbf{u}\cdot\nabla)w_t \\
&\qquad + (\mathbf{u}_t\cdot\nabla)w + ww_{tz} + w_t w_z = g_t, \\
&\mathrm{div}\,\mathbf{u}_t + w_{tz} = 0, \qquad (\mathbf{u}_t, w_t)\Big|_{\partial\Omega\times[0,T]} = 0.
\end{aligned} \tag{21}$$

Take scalar product of (21) and $(\mathbf{u}_t, w_t)$:

$$\frac{1}{2}\frac{d}{dt}\|\mathbf{u}_t\|^2 + \frac{1}{2}\frac{d}{dt}\|w_t\|^2 + \nu\|\mathbf{u}_{tx}\|^2 + \nu\|w_{tx}\|^2$$

$$+\varepsilon\nu\int\limits_\Omega (\nabla\mathbf{u})^2(\nabla\mathbf{u}_t)^2 dx + \frac{\varepsilon\nu}{2}\|[(\nabla\mathbf{u})^2]_t\|^2 + (u_{kt}\mathbf{u}_{x_k}, \mathbf{u}_t) \tag{22}$$

$$+(w_t\mathbf{u}_z, \mathbf{u}_t) + (u_{kt}w_{x_k}, w_t) + (w_t w_z, w_t) = (\mathbf{f}_t, \mathbf{u}_t) + (g_t, w_t).$$

To estimate scalar products of (22) we need the following inequalities

$$\|v\|_{10/3} \le (48)^{1/10}\|v_x\|^{3/5}\|v\|^{2/5},$$
$$\|v\|_3 \le (48)^{1/2}\|v_x\|^{1/2}\,\|v\|^{1/2}, \tag{23}$$
$$\|v\|_{8/3} \le (48)^{1/16}\|v_x\|^{3/8}\,\|v\|^{5/8},$$

being valid for functions from the Sobolev space $H_0^1(\Omega),\ \Omega \in R^3$. Then we have

$$|I_1| = |(u_{kt}\mathbf{u}_{x_k}, \mathbf{u}_t)| \le c\|\nabla\mathbf{u}\|_3\|\mathbf{u}_t\|_3^2 \le c\|\nabla\mathbf{u}\|_4\|\mathbf{u}_{tx}\|\|\mathbf{u}_t\|$$
$$\le \delta\|\mathbf{u}_{tx}\|^2 + \frac{c}{\delta}\|\nabla\mathbf{u}\|_4^2\|\mathbf{u}_t\|^2,$$

Estimate now the second scalar product. Integration by parts and the use of the incompressibility equation give

$$I_2 = (w_t\mathbf{u}_z, \mathbf{u}_t) = (\operatorname{div}\mathbf{u}_t\mathbf{u}, \mathbf{u}_t) - (w_t\mathbf{u}, \mathbf{u}_{tz}) = I_2' + I_2''.$$

Estimate each of these scalar products separately using the Hölder and Young inequalities. We get

$|I_2'| = |(\operatorname{div}\mathbf{u}_t\mathbf{u}, \mathbf{u}_t)| \le$ (use the Hölder inequality with the powers 2, 5, 10/3)

$\le \|\mathbf{u}_{tx}\|\|\mathbf{u}\|_5\|\mathbf{u}_t\|_{10/3} \le$ (due to (23)) $\le (48)^{1/10}\|\mathbf{u}_{tx}\|^{8/5}\|\mathbf{u}\|_5\|\mathbf{u}_t\|^{2/5}$

$\le$ (due to the Young inequality) $\le \delta\|\mathbf{u}_{tx}\|^2 + c_\delta\|\mathbf{u}\|_5^5\|\mathbf{u}_t\|^2 \le$ (due to (17))

$\le \delta\|\mathbf{u}_{tx}\|^2 + c_\delta(\|\nabla\mathbf{u}\|_4^4 + \|\mathbf{u}_z\|^2)\|\mathbf{u}_t\|^2,$

In the same way one gets

$$|I_2''| = |(w_t\mathbf{u}, \mathbf{u}_{tz})| \leq \text{(use the Hölder inequality with the powers 10/3, 5, 2)}$$

$$\leq c\|\mathbf{u}_{tz}\|\|\mathbf{u}\|_5\|w_t\|_{10/3} \leq \text{ (due to (23)) } \leq c\|\mathbf{u}_{tz}\|\|\mathbf{u}\|_5\|w_{tx}\|^{3/5}\|w_t\|^{2/5}$$

$$\leq \delta\|\mathbf{u}_{tz}\|^2 + c_\delta\|\mathbf{u}\|_5^2\|w_{tx}\|^{6/5}\|w_t\|^{4/5}$$

$$\leq \text{ (due to the Young inequality with the powers 5/3, 5/2)}$$

$$\leq \delta\|\mathbf{u}_{tz}\|^2 + \delta\|w_{tx}\|^2 + c_\delta\|\mathbf{u}\|_5^5\|w_t\|^2 \quad \text{(due to (17))}$$

$$\leq \delta\|\mathbf{u}_{tz}\|^2 + \delta\|w_{tx}\|^2 + c_\delta(\|\nabla\mathbf{u}\|_4^4 + \|\mathbf{u}_z\|^2)\|w_t\|^2.$$

For estimation the other two scalar products we need the following

Lemma 3. *The estimate*

$$\max_z |w|_{4,E_{xy}} \le c\|\nabla \mathbf{u}\|_4 \tag{24}$$

holds.
Proof. As before, extend w and $\mathbf{u}$ onto the whole R^3 by zero and denote the obtained functions by the same letters. Then, we get

$$|w|_{4,E_{xy}} = \left(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(\int_{-\infty}^{z} \operatorname{div}\mathbf{u}dz\right)^4 dxdy\right)^{1/4} \le \int_{-\infty}^{\infty} |\nabla \mathbf{u}|_{4,E_{xy}}dz \le c\|\nabla \mathbf{u}\|_4.$$

The statement of the lemma follows directly from the last inequality. Q.E.D.

Estimate now the scalar product I_3:

$$I_3 = (u_{kt}w_{x_k}, w_t) = -(\operatorname{div}\mathbf{u}_t w, w_t) - (u_{kt}w, w_{tx_k})$$

$$= (w_{tz}w, w_t) - (u_{kt}w, w_{tx_k}) = I_3' + I_3'';$$

Obtain a priori estimates for I_3' and I_3'' separately. One has

$$|I_3''| = |(u_{kt}w, w_{tx_k})| \le \int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |u_{kt}w, w_{tx_k}|dxdy\right)dz$$

$$\le c\|\nabla \mathbf{u}\|_4 \int_{-\infty}^{\infty} |\mathbf{u}_t|_{4,E_{xy}}|w_{tx}|_{2,E_{xy}}dz$$

$$\le c\|\nabla \mathbf{u}\|_4 \int_{-\infty}^{\infty} |\nabla \mathbf{u}_t|_{2,E_{xy}}^{1/2}|\mathbf{u}_t|_{2,E_{xy}}^{1/2}|w_{tx}|_{2,E_{xy}}dz$$

$$\le c\|\nabla \mathbf{u}\|_4\|\nabla \mathbf{u}_t\|^{1/2}\|\mathbf{u}_t\|^{1/2}\|w_{tx}\| \le \delta\|w_{tx}\|^2 + c_\delta\|\nabla \mathbf{u}\|_4^2\|\nabla \mathbf{u}_t\|\|\mathbf{u}_t\|$$

$$\le \delta\|w_{tx}\|^2 + \delta\|\nabla \mathbf{u}_t\|^2 + c_\delta\|\nabla \mathbf{u}\|_4^4\|\mathbf{u}_t\|^2,$$

$$|I_3'| = |(w_t, w_z, w_t)| = |(\operatorname{div}\mathbf{u}, w_t^2)| \le \|\nabla\mathbf{u}\|_4 \|w_t\|_{8/3}^2$$

$$\le c\|\nabla\mathbf{u}\|_4 \|w_{tx}\|^{3/4}\|w_t\|^{5/4} \le \delta\|w_{tx}\|^2 + c_\delta\|\nabla\mathbf{u}\|_4^{8/5}\|w_t\|^2.$$

Finally, $I_4 = I_3'$, so I_4 is estimated from above as I_3'. Substituting the above inequalities into (22) with appropriate δ and estimating the right-hand side of (22) by the obvious way, we get

$$\begin{aligned}
&\frac{d}{dt}(\|\mathbf{u}_t\|^2 + \|w_t\|^2) + \nu\|\mathbf{u}_{tx}\|^2 + \nu\|w_{tx}\|^2 \\
&+\varepsilon\nu\int_\Omega (\nabla\mathbf{u})^2(\nabla\mathbf{u}_t)^2 dx + \frac{\varepsilon\nu}{2}\|[(\nabla\mathbf{u})^2]_t\|^2 \qquad (25)\\
&\le c\left(\|\mathbf{f}_t\|_{-1}^2 + \|g_t\|_{-1}^2 + (\|\nabla\mathbf{u}\|_4^4 + \|w_x\|^2)(\|\mathbf{u}_t\|^2 + \|w_t\|^2)\right).
\end{aligned}$$

Using the Gronwall inequality, from (25) it follows

$$\begin{aligned}
&\max_{0\le t\le T}(\|\mathbf{u}_t(t)\|^2 + \|w_t(t)\|^2) + \int_0^T (\|\mathbf{u}_{tx}\|^2 + \|w_{tx}\|^2)dt \\
&\le \left(\|\mathbf{u}_t(0)\|^2 + \|w_t(0)\|^2 + \int_0^T (\|\mathbf{f}_t\|_{-1}^2 + \|g_t\|_{-1}^2)dt\right) \qquad (26)\\
&\times \exp\left(c\int_0^T (\|\nabla\mathbf{u}\|_4^4 + \|w_x\|^2)dt\right).
\end{aligned}$$

Introduce the space $\mathbf{V}$ of divergence free vector functions $(\mathbf{u}, w)$ vanishing on $\partial\Omega \times [0,T]$ with the norm

$$\|(\mathbf{u},w)\|_V = \left(\int_0^T \left(\|\mathbf{u}_x\|^2 + \|w_x\|^2 + \|\mathbf{u}_t\|^2 + \|w_t\|^2\right) dt \right)^{1/2} + \left(\int_0^T \|\nabla \mathbf{u}\|_4^4 dt \right)^{1/4}$$

and define a solution to (14) as a vector function $(\mathbf{u}, w) \in \mathbf{V}$ being equal to $(\mathbf{u}_0, w_0)$ for $t = 0$ and satisfying the following identity

$$\begin{aligned} \int_0^T \Big(&(\mathbf{u}_t, \mathbf{v}) + \nu(\mathbf{u}_x, \mathbf{v}_x) + \nu\varepsilon(|\nabla \mathbf{u}|^2 \nabla \mathbf{u}, \nabla \mathbf{v}) + \nu(w_x, h_x) \\ &+((\mathbf{u}\cdot\nabla)\mathbf{u}, \mathbf{v}) + ((\mathbf{u}\cdot\nabla)w, h) + (ww_z, h) \\ &-(\mathbf{f}, \mathbf{v}) - (g, h)\Big) dt = 0 \quad \forall (\mathbf{v}, h) \in \mathbf{V}. \end{aligned} \tag{27}$$

Using the Galerkin method and estimate (26), it is not difficult to prove that a solution to (27) exists and is unique and the norm $\|\mathbf{u}_x\| + \|w_x\|$ is continuous in time.

Thus, we have proved

Theorem 2. *For any initial condition*

$$\mathbf{u}_0 \in \mathbf{H}^2 \cap \mathbf{H}_0^1, \; w_0 \in H^2 \cap H_0^1, \; div\, \mathbf{u} + w_z = 0,$$

right-hand sides $\mathbf{f}$ *and* g *such that* $\int_0^T (\|\mathbf{f}_t\|_{-1}^2 + \|g_t\|_{-1}^2) dt < \infty,$

any $\nu > 0$, $\varepsilon > 0$ *and arbitrary time interval* $[0,T]$ *there exists a solution to (14) "in the large", i.e. there exists a unique solution* $(\mathbf{u}, w) \in V$ *to (14) satisfying (27) and the norm* $\|\mathbf{u}_x\|^2 + \|w_x\|^2$ *is continuous in time on* $[0,T]$.